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INVARIANT AFFINOR METRIC STRUCTURES ON LIE GROUPS

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Abstract: We introduce the class of special metric structures on Lie groups which are connected with the radical of a fixed 1-form on a Lie group. These structures are called affinor metric structures. We introduce and study some special classes of invariant affinor metric structures and generalize many results of the theory of contact metric structures on Lie groups.

Keywords: left-invariant metric, affinor, radical of a 1-form, Lie group

Introduction

We introduce and study some special types of metrics on Lie groups that are connected with a given 1-form and its radical. These metric structures are called *affinor metric structures*. The simplest partial case of affinor metric structures are contact metric structures [1]. Many of the notions and results of the article have analogs in the theory of contact structures. But most of the results of Section 5 cannot be applied to contact metric structures.

In Section 1, we study the properties of the radical of 1-forms on Lie groups. In Sections 2–4, we introduce and study various metric structures connected with the radical of a given 1-form. In Section 5, we consider the connection between the affinor metric structures and the complex structures on Lie groups.

In this article, most of the results previously known only for 1-forms with one-dimensional radical (contact form) are proved for the 1-forms with radical of arbitrary dimension. We also obtain some topological and geometric facts that describe the structure of Lie algebras and Lie groups with affinor structure.

By an affinor metric structure we mean a triple (α, Φ, β), where α is a 1-form with nontrivial radical, β is a symmetric 2-form whose restriction to the radical of α is a Riemannian metric and Φ is a field of linear operators on the Lie group making the form $d\alpha$ into a Riemannian metric on the subspace complementary to the radical of α. The form β is called the *radical metric* and studied separately in Section 2. The field Φ is called the *affinor* and is a key notion of the theory of affinor metric structures.

§ 1. The Radical of Linear Forms

Suppose that G is a connected Lie group of dimension n, while $\mathfrak{g}$ is its Lie algebra, α is a left-invariant 1-form on G, and β is a left-invariant 2-form on G. Unless otherwise specified, we assume all forms on Lie groups left-invariant throughout the sequel.

Definition 1.1. By the *radical of a 2-form* β we mean the maximal subset $\operatorname{rad}\beta \subset \mathfrak{g}$ such that $\beta(X, Y) = 0$ for all $X \in \operatorname{rad}\beta$ and $Y \in \mathfrak{g}$. By the *radical* of a 1-form α we mean the radical of the exterior derivative $d\alpha$ of α.

From the definition it is immediate that the radical of a 1-form on a Lie group G is a vector space, always including the center of the Lie algebra of G. The 2-forms with zero radical are called *symplectic* and studied in [2].

Example 1. Let G be a Lie group of dimension $2n + 1$ and let α be a left-invariant 1-form on G having the property $(d\alpha)^n \wedge \alpha \neq 0$. These are called *contact forms* and studied in [3]. In [4] it is proved that every left-invariant contact structure has one-dimensional radical.

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EXAMPLE 2. Let H be an odd-dimensional Lie group with left-invariant contact form $\hat{\alpha}$ and let A be a commutative Lie group of odd dimension. Consider the group $G = H \times A$. Extend the form $\hat{\alpha}$ to the left-invariant form α on G by putting $\alpha \equiv 0$ on A. The group G is even-dimensional, and so is the radical α, since α is formed by the direct sum of the Lie algebra of A and the one-dimensional radical of $\hat{\alpha}$.

Proposition 1.2. *Let α be a left-invariant 1-form on a Lie group G. Then $\operatorname{rad}\alpha$ is a subalgebra of the Lie algebra of G.*

PROOF. Let $\mathfrak{g}$ be the Lie algebra of G. Given X and Y in $\operatorname{rad}\alpha$ and Z in $\mathfrak{g}$, we have

$$d\alpha([X,Y],Z) = -(1/2)\alpha([[X,Y],Z]).$$

Using Jacobi's identity, we infer

$$\alpha([[X,Y],Z]) = -\alpha([[Y,Z],X]) - \alpha([[Z,X],Y]) = 2d\alpha([Y,Z],X) + 2d\alpha([Z,X],Y).$$

By the definition of the radical of a 1-form, the right-hand side of the last equality is zero. Hence, $d\alpha([X,Y],Z) = 0$; consequently, $[X,Y] \in \operatorname{rad}\alpha$.

Denote by $\mathfrak{r}$ the radical of α and by R, the connected subgroup generated by the subalgebra $\mathfrak{r}$. We call R the *radical subgroup* and suppose that the subgroup R acts at the 1-form α by the rule $\operatorname{Ad}^*_R\alpha$, i.e., $\operatorname{Ad}^*_r \alpha(X) = \alpha(\operatorname{Ad}_r X)$ for all $r \in R$ and $X \in \mathfrak{g}$.

Proposition 1.3. *The radical subgroup of a left-invariant 1-form α on a Lie group G is closed and coincides with the connected component of the unity of the isotropy subgroup of the form α with respect to the relatively adjoint action of G.*

PROOF. Let H be the isotropy subgroup of α and let $\mathfrak{h}$ be its Lie algebra. If a vector X belongs to $\mathfrak{h}$ then the integral curve $h(t)$ issuing from the unity of G in the direction of X lies in H. Hence,

$$\frac{d}{dt}\Big|_{t=0} \operatorname{Ad}^*_{h(t)} \alpha(Y) = \alpha([X,Y]) = 0 \quad \text{for all } Y \in \mathfrak{g},$$

i.e., $X \in \mathfrak{r}$.

Conversely, if X belongs to $\mathfrak{r}$ and $r(t)$ is the integral curve issuing from the unity of G in the direction of X then

$$\frac{d}{dt} \operatorname{Ad}^*_{r(t)} \alpha(Y) = \alpha(\operatorname{Ad}_{r(t)}[X,Y]) = \alpha([\operatorname{Ad}_{r(t)} X, \operatorname{Ad}_{r(t)} Y]) = \alpha([X, \operatorname{Ad}_{r(t)} Y]) = 0$$

for all $Y \in \mathfrak{g}$, i.e., the curve $r(t)$ lies in H, and, hence, $X \in \mathfrak{h}$. Thus, $\mathfrak{h} = \mathfrak{r}$.

Since $\mathfrak{h} = \mathfrak{r}$, the connected component of the unity of H is generated by the subalgebra $\mathfrak{r}$ and, hence, coincides with R, and the subgroup R is closed because the connected component of the unity of every Lie group is always closed.

Lemma 1.4. *The dimension of the radical of every left-invariant 1-form on a Lie group of odd dimension is greater than or equal to 1.*

PROOF. Suppose that G is a Lie group of dimension $2n+1$, $\mathfrak{g}$ is the Lie algebra of G, and α is a left-invariant 1-form on G. It is known that, in a fixed basis for the Lie algebra, the exterior 2-form $d\alpha$ is represented by a skew-symmetric matrix of order $2n+1$. Since the characteristic polynomial of the matrix has odd degree, it has at least one real root. Consequently, the matrix of $d\alpha$ has at least one eigenvector with a real eigenvalue λ. Introduce some new basis of the Lie algebra $\mathfrak{g}, e_1, \dots, e_{2n+1}$ so that e_{2n+1} be an eigenvector with eigenvalue λ. In this basis the matrix of $d\alpha$ takes the form

$$\begin{bmatrix} a_{11} & \dots & a_{1,2n} & 0 \\ \dots & \dots & \dots & 0 \\ a_{2n+1,1} & \dots & a_{2n+1,2n} & \lambda \end{bmatrix}.$$

The skew-symmetry of this matrix implies that $\lambda = 0$ and $a_{2n+1,k} = 0$ for all $k = 1, \dots, 2n$. Consequently, e_{2n+1} generates a one-dimensional subspace in the radical of α.

Theorem 1.5. *Let α be a left-invariant nonclosed* 1*-form on a Lie group G of dimension n. If n is even then the dimension of* $\operatorname{rad}\alpha$ *is also even and* $0 \le \dim(\operatorname{rad}\alpha) \le n-2$, *and if n is odd then the dimension of* $\operatorname{rad}\alpha$ *is also odd and* $1 \le \dim(\operatorname{rad}\alpha) \le n-2$.

PROOF. Let $\mathfrak{g}$ be the Lie group of G. Suppose that n is even. If α has a nontrivial radical on $\mathfrak{g}$ then choose a one-dimensional subspace V_1 in the radical and denote by W_1 the subspace complementary to V_1 in $\mathfrak{g}$ such that the restriction of α to W_1 is not identically zero. Such a subspace may always be chosen since a linear form cannot be identically zero on $\mathfrak{g} \setminus V_1$. Furthermore, $d\alpha \not\equiv 0$ on W_1, since otherwise $d\alpha \equiv 0$ on $\mathfrak{g}$, which contradicts the nonclosedness of α.

Since W_1 has odd dimension, by Lemma 1.4 it includes a one-dimensional subspace V_2 in the radical of α. Clearly, the two-dimensional subspace $\{V_1, V_2\}$ is in the radical of α. Denote by W_2 the subspace complementary to V_2 in W_1 such that the restriction of α to W_2 is not identically zero. If α is nondegenerate on W_2 then the assertion is proved. If W_2 includes a one-dimensional subspace of $\operatorname{rad}\alpha$ then repeat the previous step. Continuing by induction, we infer that the dimension of $\operatorname{rad}\alpha$ is even. Similar arguments in the case when n is odd show that the dimension $\operatorname{rad}\alpha$ is odd in this case.

If $\dim(\operatorname{rad}\alpha) = n-1$ then from Definition 1.1 it follows that $d\alpha \equiv 0$ on G. This contradicts the nonclosedness of α. Thus, the proof of the theorem is complete.

REMARK 1.6. In [1] it was proved that insoluble unimodular Lie groups do not admit nondegenerate closed exterior 2-forms. Therefore, on an insoluble unimodular Lie group of even dimension, the inequality

$$1 \le \dim(\operatorname{rad}\alpha) \le n-1$$

holds for every 1-form α. It also follows from Theorem 1.5 that, on a two-dimensional Lie group, the radical of every 1-form is trivial; i.e., either it is zero or coincides with the Lie algebra.

The following result enables us to determine the dimension of the radical of a 1-form on semisimple Lie groups:

Theorem 1.7. *Let G be a semisimple Lie group and let α be a left-invariant* 1*-form on G. Then the radical of α coincides with the greatest subspace V such that, for every X in V, the operator* ad_X *is skew-symmetric with respect to the* 2*-form* $d\alpha$.

PROOF. If $X \in \operatorname{rad}\alpha$ then, for all Y and Z in $\mathfrak{g}$, we have

$$\begin{aligned} 6\, d^2\alpha(X,Y,Z) &= -d\alpha([X,Y],Z) + d\alpha([X,Z],Y) - d\alpha([Y,Z],X) \\ &= d\alpha(\operatorname{ad}_X Z, Y) + d\alpha(Z, \operatorname{ad}_X Y) = 0, \end{aligned}$$

i.e., $X \in V$.

Conversely, if $X \in V$ then $d\alpha(\operatorname{ad}_X Y, Z) = -d\alpha(Y, \operatorname{ad}_X Z)$ for all Y and Z in $\mathfrak{g}$, whence

$$6\, d^2\alpha(X,Y,Z) = d\alpha(X,[Y,Z]) = 0.$$

Since, for semisimple Lie groups, the first derived ideal coincides with the entire Lie algebra, each vector field in $\mathfrak{g}$ may be represented as the Lie bracket of two other vector fields. Thus, $d\alpha(X,Y) = 0$ for all Y in $\mathfrak{g}$, and $X \in \operatorname{rad}\alpha$.

Theorem 1.7 gives a simple algorithm for calculating the dimension of the radical of a 1-form α on a semisimple Lie group G. If we fix a basis $E_1, \dots, E_n$, $n = \dim G$, of the Lie algebra of G then the dimension of the radical of α is equal to the number of the basis vectors for which the operators ad_{E_i} are skew-symmetric in $d\alpha$.

If we assume that the radical subgroup R acts on G by right multiplication and is a normal subgroup in G then we may consider the fibration $G \xrightarrow{\pi} G/R$ with fiber R and the natural projection π.

Theorem 1.8. *Let G be a connected simply connected Lie group and let R be the normal radical subgroup of a left-invariant 1-form α on G. Then the fibration $G \xrightarrow{\pi} G/R$ is trivial if and only if G is isomorphic to the semidirect product $G/R \rtimes R$.*

PROOF. Denote $G \xrightarrow{\pi} G/R$ by P. If P is trivial then there exists a homeomorphism between G and $G/R \times R$ and so the group $G/R \rtimes R$ is simply connected. The triviality of P implies that P admits a global section Σ (see [5, Vol. 1, Chapter 2]).

Let Φ be a homomorphism from G/R into the group of automorphisms of the semigroup R of the form

$$\Phi_a(h) = \Sigma(a)h\Sigma(a)^{-1}, \quad a \in G/R,\ h \in R.$$

Since R is a normal subgroup, Φ_a acts on R invariantly for every $a \in G/R$. This implies that the radical $\mathfrak{r}$ is an ideal in $\mathfrak{g}$ and the Lie algebra $\mathfrak{g}$ is isomorphic to the semidirect product of $\mathfrak{g}/\mathfrak{r}$ and $\mathfrak{r}$. The isomorphism of the Lie algebras induces the group isomorphism between G and $G/R \rtimes R$.

Conversely, if G and $G/R \rtimes R$ are isomorphic then the isomorphism of these groups is, in particular, a homeomorphism of the topological spaces G and $G/R \times R$ and so the fibration P is trivial.

§ 2. A Metric of the Radical

Suppose that G is a connected Lie group, α is a left-invariant 1-form on G, $\mathfrak{r}$ is its radical, and R is the radical subgroup.

DEFINITION 2.1. By a *metric of the radical* $\mathfrak{r}$ we mean a left-invariant symmetric nonnegative 2-form β possessing the following properties:

(1) β is nondegenerate on $\mathfrak{r}$;

(2) β has radical of maximal dimension, i.e., the Lie algebra of G may be represented as the direct sum of $\mathfrak{g} = \mathfrak{r} \oplus \operatorname{rad}\beta$.

It can be seen from the definition that the restriction of β to $\mathfrak{r}$ defines a Riemannian metric on $\mathfrak{r}$.

EXAMPLE 1. Suppose that a group G has the form $H \times R$, where H is a symplectic Lie group with left-invariant symplectic structure Ω and R is a commutative Lie group. Let g_0 be the standard left-invariant Euclidean metric on G in some fixed basis of the Lie algebra $\mathfrak{g}$ and let α be a left-invariant 1-form on G such that the restriction of $d\alpha$ to H coincides with Ω. Since $d\alpha$ is nondegenerate on H and R is commutative, $d\alpha(X, Y) = -(1/2)\alpha([X, Y]) = 0$ for all $X \in \mathfrak{r}$ and $Y \in \mathfrak{g}$, i.e., $\operatorname{rad}\alpha = \mathfrak{r}$, and as the metric of the radical we may take the form $g_0 \circ d\pi$, where π is the projection of G onto R along H.

EXAMPLE 2. Suppose that G is an insoluble Lie group, R is a maximal soluble subgroup in G admitting a left-invariant exact symplectic structure Ω, and α is a left-invariant 1-form on G such that the restriction of $d\alpha$ to R coincides with Ω. By Levi's Theorem (for example, see [6]), G may be represented as $G = S \rtimes R$, where S is a semisimple Lie group. If, for every $X \in S$, the image of the Lie algebra $\mathfrak{g}$ under the action of ad_X lies in the kernel of α then $\operatorname{rad}\alpha = \mathfrak{s}$ and $\beta = -B$, where B is the Killing–Cartan form, defines a left-invariant metric of the radical on G.

Denote the radical of the metric of the radical β by D. It may be considered as a left-invariant distribution on G. Definition 2.1 implies that $\mathfrak{g} = D \oplus \mathfrak{r}$. Since $\mathfrak{r}$ is a subalgebra, Ad_R acts on $\mathfrak{r}$ invariantly. If Ad_R acts invariantly on D then the Lie algebra of G is reducible in the sense of Nomizu.

Proposition 2.2. *If the metric of the radical is Ad_R-invariant then the distribution D is invariant under the relatively adjoint action of the radical subgroup.*

PROOF. Suppose that $X \in D$ and $Y \in \mathfrak{g}$. If the metric of the radical β is Ad_R-invariant then, given $h \in R$, we have

$$\beta(\operatorname{Ad}_h X, Y) = \beta(X, \operatorname{Ad}_{h^{-1}} Y) = 0,$$

i.e., $\operatorname{Ad}_h X \in D$.

Proposition 2.3. *Let β be a left-invariant metric of the radical $\mathfrak{r}$ on a Lie group G and let $D = \operatorname{rad}\beta$. If the radical subgroup R is a compact subgroup in G and the projection π of the Lie algebra $\mathfrak{g}$ onto the subalgebra $\mathfrak{r}$ along D commutes with the adjoint action of the radical subgroup then the distribution D is invariant under the adjoint action of R.*

PROOF. Applying the operation of averaging of the form β over R, we obtain an Ad_R-invariant form β_1. Since $\mathrm{Ad}_h \circ d\pi = d\pi \circ \mathrm{Ad}_h$ for every $h \in R$ and $\pi X = 0$ for every X in D, the form $\beta_2 = \beta_1 \circ d\pi$ is an Ad_R-invariant metric of the radical $\mathfrak{r}$ with radical D. By Proposition 2.2, we see that D is invariant under the adjoint action of R.

REMARK 2.4. If the radical subgroup R is a maximal torus in G and the projection $\pi : \mathfrak{g} \mapsto \mathfrak{r}$ along the radical D of some left-invariant metric of the radical $\mathfrak{r}$ commutes with the adjoint action of the maximal torus then, since every torus is a commutative compact group, the distribution D is invariant under the adjoint action of the maximal torus R.

Denote the fibration $G \overset{\pi}{\to} G/R$ of Section 1 by P.

Theorem 2.5. *Suppose that G is a connected Lie group, α is a left-invariant 1-form on G with radical $\mathfrak{r}$, β is a left-invariant metric of the radical, D is the radical of the form β, the radical subgroup R is a maximal torus in G, and the projection $\pi : \mathfrak{g} \mapsto \mathfrak{r}$ along D commutes with the adjoint action of the subgroup of the radical. Then*

(1) *the distribution D is invariant under the adjoint action of the radical subgroup and is a connection of P;*

(2) *the connection form of D is as follows:*

$$\omega(X) = \sum_{i=1}^{m} \beta(X, E_i)E_i,$$

where $E_1, \dots, E_m$ is a fixed basis of the radical $\mathfrak{r}$ orthonormal with respect to the metric β and m is the dimension of $\mathfrak{r}$, $X \in \mathfrak{g}$;

(3) *D is flat (i.e., has zero form of the curvature of the connection) if and only if D is involutive.*

PROOF. (1) By Remark 2.4, Ad_R acts on D invariantly. This implies that the distribution D is bi-invariant under the adjoint action of R and, in particular, invariant under right translations by the elements of the radical subgroup. Since it is differentiable and is the complement to the radical $\mathfrak{r}$, it follows that D is a connection for P.

(2) Suppose that $X = \sum_{i=1}^{m} x_i E_i \in \mathfrak{r}$. We have

$$\omega(X) = \sum_{i=1}^{m} x_i \omega(E_i) = \sum_{i=1}^{m} x_i E_i = X.$$

Since a maximal torus is a commutative subgroup, for all h in R and Y in $\mathfrak{r}$, we will have $\mathrm{Ad}_h Y(e) = Y(e)$ or $dL_h Y(e) = dR_h Y(e)$, where e is the unity of G. Given h in R, we have

$$R_h^* \omega(X(e)) = \sum_{i=1}^{m} \beta(dR_h X(e), E_i(h))E_i(h)$$

$$= \sum_{i=1}^{m} \beta(\mathrm{Ad}_h X(h), E_i(h)) \mathrm{Ad}_{h^{-1}} E_i(h) = \mathrm{Ad}_{h^{-1}}(\omega(X(h))),$$

i.e., the form ω satisfies all axioms of a connection form.

Show that ω does not depend on the choice of a basis. Let $E'_1, \dots, E'_m$ be another orthonormal basis for the radical and let A be the orthogonal transformation of this basis to a basis $E_1, \dots, E_m$. We have

$$\begin{aligned}\omega(X) &= \sum_{i=1}^{m} \beta(X, E_i)E_i = \sum_{i=1}^{m} \sum_{k,l} \beta\big(X, a_i^k E'_k\big) a_i^l E'_l \\ &= \sum_{i=1}^{m} \sum_{k,l} a_i^k a_i^l \beta(X, E'_k)E'_l = \sum_{k=1}^{m} \beta(X, E'_k)E'_l = \omega'(X).\end{aligned}$$

(3) Let Ω be the curvature form of D. Using the structural equation of [5, Chapter 2] and the commutativity of the radical $\mathfrak{r}$, we infer

$$\Omega(X,Y) = (1/2)[\omega(X), \omega(Y)] + d\omega(X,Y) = d\omega(X,Y) = -\omega([X,Y])$$

for all X and Y in $\mathfrak{g}$.

Since the radical $\mathfrak{r}$ is commutative, $\omega([X,Y]) = 0$ for all X and Y in $\mathfrak{r}$. The distribution D is the kernel of the connection form ω and is invariant under Ad_R; therefore, $[X,Y] = \mathrm{ad}_X Y \in D$ for all $X \in D$ and $Y \in \mathfrak{r}$, and so $\omega([X,Y]) = 0$. If D is involutive then $\omega([X,Y]) = 0$ for all X and Y in D. Thus, $\Omega(X,Y) = 0$ for all X and Y in $\mathfrak{g}$.

Conversely, if $\Omega \equiv 0$ and $X, Y \in D$ then

$$\omega([X,Y]) = -\Omega(X,Y) = 0,$$

i.e., $[X,Y] \in D$.

Remark 2.6. The assertions of Theorem 2.5 hold also in the following two cases:

(1) the radical subgroup R is commutative (not necessarily compact), and the radical metric is Ad_R-invariant;

(2) the radical subgroup is an arbitrary subgroup in G and every vector field in $\mathfrak{g}$ is Ad_R-invariant.

Proposition 2.7. *Let α be a left-invariant nonclosed 1-form on a Lie group G and let D be the radical of a left-invariant metric of the radical of α. If D is a subalgebra of the Lie algebra of G then D cannot lie in the kernel of the form α.*

Proof. If D is not a subalgebra in $\mathfrak{g}$ then D is involutive and $\mathfrak{g} = D \oplus \mathrm{rad}\,\alpha$. If $D \subset \ker\alpha$ then we infer $d\alpha(X,Y) = -(1/2)\alpha([X,Y]) = 0$ for all X and Y in D. Thus, $d\alpha \equiv 0$ on D, and hence $d\alpha \equiv 0$ on $\mathfrak{g}$, which contradicts the nonclosedness of α.

Remark 2.8. If α is a left-invariant contact structure on a Lie group G then every left-invariant metric of the radical of a form α is proportional to the 2-form $\alpha \otimes \alpha$, and its radical coincides with the kernel of the form α. Consequently, in the contact case, the distribution D cannot be a subalgebra.

Suppose that the distribution D is involutive and invariant under the adjoint action of the radical subgroup. In this case D is a subalgebra in $\mathfrak{g}$, and the Lie algebra $\mathfrak{g}$ is isomorphic to the semidirect product of the subalgebra D and the radical $\mathfrak{r}$. Denote by H the connected subgroup generated by D. It was proved in [7] that an isomorphism of Lie algebras implies a local isomorphism of the Lie groups. Consequently, there exists a local isomorphism from G onto the semidirect product of the subgroups H and R. This isomorphism defines a local homeomorphism of the topological spaces G and $H \times R$.

If we assume that the radical subgroup acts on G by right multiplication then we may introduce the fibration $G \xrightarrow{\pi} H$ with fiber R and projection π. Denote this fibration by L. If the radical subgroup R is commutative then D is a flat connection of L with connection form ω of item (2) of Theorem 2.5. Note that, in the fibration L, we require no longer that the radical subgroup be commutative and compact. Moreover, if the metric of the radical is Ad_R-invariant then we may remove the requirement of the Ad_R-invariance of D, since it follows from Proposition 2.2. From Proposition 2.7 and Remark 2.8 it follows that the fibration L cannot be constructed for left-invariant contact structures.

§ 3. Affinor Metric Structures

Suppose that G is a connected Lie group, α is a left-invariant 1-form, β is a left-invariant metric of the radical $\mathfrak{r}$, and D is the radical of β.

Definition 3.1. An *affinor of a 1-form* α is a field of endomorphisms Φ of the Lie algebra $\mathfrak{g}$ of G having the following properties:

(1) $\Phi X \in D$ for all X in $\mathfrak{g}$ and $\Phi Y = 0$ for all Y in $\mathfrak{r}$;

(2) $d\alpha(\Phi X, \Phi Y) = d\alpha(X, Y)$ for all X and Y in $\mathfrak{g}$;

(3) $\Phi^2 X = -X + \omega(X)$ for all X in $\mathfrak{g}$, where ω is the connection form of item (2) of Theorem 2.5;

(4) a 2-form $d\alpha(X, \Phi Y)$ is positive definite for all X and Y in D.

An affinor Φ is called *left-invariant* if Φ commutes with left translations on G and *bi-invariant* if Φ commutes with both left and right translations on G.

From (3) it follows that if $X \in D$ then $\Phi^2 X = -X$; properties (2) and (3) imply that $d\alpha(\Phi X, Y) = -d\alpha(X, \Phi Y)$ for all X and Y in $\mathfrak{g}$, and (1) and (4) imply that the radical of $D\alpha(X, \Phi Y)$ coincides with $\operatorname{rad}\alpha$ and its restriction to D is a Riemannian metric.

Definition 3.2. An *affinor metric structure g on a Lie group* G is a triple (α, Φ, β) such that

$$g(X, Y) = d\alpha(X, \Phi Y) + \beta(X, Y) \quad \text{for all } X \text{ and } Y \text{ in } \mathfrak{g},$$

where α is a 1-form with radical $\mathfrak{r}$, Φ is the affinor of α, and β is the metric of a radical $\mathfrak{r}$.

An affinor metric structure is called *left-invariant* if the tensor fields α, ϕ, and β are left-invariant and *bi-invariant* if these tensor fields are bi-invariant.

It is straightforward from Definition 3.2 that the distribution D and the radical $\mathfrak{r}$ are orthogonal with respect to the affinor metric. Thus, if the Lie group is endowed with an affinor metric structure then D may be defined uniquely as the orthogonal complement to the radical of α.

Suppose that $d\alpha_\Phi$ is the restriction of $d\alpha(X, \Phi Y)$ to D, the distribution D is involutive, H is the subgroup generated by D, and L is the fibration $G \xrightarrow{\pi} H$ of Section 2. Then $d\alpha_\Phi$ is a metric on the base of the fibration L and the metric of the radical β_h, $h \in H$, defines some metric in each fiber over a point $h \in H$. Thus, we may apply all notions and results of [8] to affinor metric structures.

Example 1. Let G be a connected Lie group of dimension $2n+1$ and let α be a left-invariant contact form on G. The radical of α has dimension 1 and is generated by a vector field X_0. Introduce a metric of the radical β by setting $\beta(X, Y) = \alpha(X)\alpha(Y)$ for all X and Y in $\mathfrak{g}$. In this case the distribution D coincides with the kernel of α and the left-invariant affinor metric structure takes the form

$$g(X, Y) = d\alpha(X, \Phi Y) + \alpha(X)\alpha(Y),$$

where Φ is an affinor of α such that $\Phi X_0 = 0$. These affinor metric structures are called *contact metric structures*; they are studied in detail in [1].

Example 2. Let G be a Lie group of the form $G = H \times R$, where H is a nilpotent symplectic Lie group with left-invariant symplectic structure ω such that the operator $\operatorname{ad}_X \circ \operatorname{ad}_Y$ is nilpotent for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ and let R be a semisimple Lie group with nontrivial center. Let α be a left-invariant 1-form on G such that the restriction of $d\alpha$ to H coincides with Ω and $\operatorname{rad}\alpha = \mathfrak{r}$. The Killing–Cartan form B degenerates on H and is nondegenerate and negative definite on R. Since the trace of any nilpotent operator is equal to zero, $\operatorname{rad} B = \mathfrak{h}$. As the metric of the radical, take the form $\beta = -B$. In this case D coincides with $\mathfrak{h}$. Let J be a left-invariant almost complex structure on H preserving the symplectic structure Ω and let X_0 be a fixed vector field lying in the center of the Lie algebra $\mathfrak{r}$. Introduce a field of linear operators Φ on G as follows:

$$\Phi X = \begin{cases} JX & \text{if } X \in \mathfrak{h}, \\ \operatorname{ad}_{X_0} X & \text{if } X \in \mathfrak{r}. \end{cases}$$

It is easy to check that Φ is an affinor for α.

Thus, the corresponding left-invariant affinor metric structure takes the form

$$g(X, Y) = d\alpha(X, \Phi Y) - B(X, Y).$$

Proposition 3.3. *A left-invariant affinor metric structure $g = (\alpha, \Phi, \beta)$ is Ad_R-invariant if and only if $\mathrm{Ad}_h \circ \Phi = \Phi \circ \mathrm{Ad}_h$ and $\mathrm{Ad}_h^* \beta = \beta$ for every h in the radical subgroup R.*

PROOF. Using the fact that the radical subgroup is a connected component of the isotropy subgroup of α, we infer

$$g(\mathrm{Ad}_h X, \mathrm{Ad}_h Y) = d\alpha(\mathrm{Ad}_h X, \Phi \circ \mathrm{Ad}_h Y) + \beta(\mathrm{Ad}_h X, \mathrm{Ad}_h Y)$$
$$= d\alpha(\mathrm{Ad}_h X, \mathrm{Ad}_h \circ \Phi Y) + \beta(X, Y) = -(1/2)\alpha(\mathrm{Ad}_h[X, \Phi Y]) + \beta(X, Y)$$
$$= -(1/2)\alpha([X, \Phi Y]) + \beta(X, Y) = g(X, Y)$$

for all h in R and X and Y in $\mathfrak{g}$.

Conversely, if an affinor metric structure G is Ad_R-invariant then the metrics $d\alpha_\Phi$ and β are also Ad_R-invariant, since they are restrictions of g. Given h in R and X, Y in D, we have

$$d\alpha(X, \mathrm{Ad}_{h^{-1}} \circ \Phi \circ \mathrm{Ad}_h Y) = d\alpha(\mathrm{Ad}_h X, \Phi \circ \mathrm{Ad}_h Y) = d\alpha(X, \Phi Y),$$

i.e., $\mathrm{Ad}_h \circ \Phi = \Phi \circ \mathrm{Ad}_h$.

Proposition 3.4. *Let R be the radical subgroup of a left-invariant 1-form α and let $g = (\alpha, \Phi, \beta)$ be an Ad_R-invariant affinor metric structure. Then D is invariant under the adjoint action of R.*

PROOF. For all h in R and X and Y in $\mathfrak{g}$, we have

$$\beta(\mathrm{Ad}_h X, \mathrm{Ad}_h Y) = g(\mathrm{Ad}_h X, \mathrm{Ad}_h Y) - d\alpha(\mathrm{Ad}_h X, \Phi \circ \mathrm{Ad}_h Y)$$
$$= g(X, Y) - d\alpha(\mathrm{Ad}_h X, \mathrm{Ad}_h \circ \Phi Y) = g(X, Y) - d\alpha(X, \Phi Y) = \beta(X, Y).$$

The assertion is now immediate from Proposition 2.2.

Proposition 3.4 makes it possible to describe the structure of the Lie algebras of Lie groups admitting Ad_R-invariant affinor metric structures. If the distribution D on such a Lie group is involutive, its Lie algebra is the orthogonal sum of D and the radical $\mathfrak{r}$ is reducible in the sense of Nomizu. If D is involutive then its Lie algebra is the semidirect product of D and $\mathfrak{r}$.

In Section 2 we show that if D is a subalgebra then α cannot be a contact structure. Therefore, the following theorem holds only for left-invariant noncontact 1-forms.

Theorem 3.5. *Suppose that R is the radical subgroup of a left-invariant noncontact 1-form α, $g = (\alpha, \Phi, \beta)$ is an Ad_R-invariant affinor metric structure on the fibration L, and ∇ is the Levi-Civita connection of g. Then*

(1) the subalgebra $D = \mathrm{rad}\, \beta$ is a connection for L; in the case when the radical subgroup is commutative, the connection D is flat and the connection form is as follows:

$$\omega(X) = \sum_{i=1}^{m} g(X, E_i) E_i,$$

where $E_1, \dots, E_m$ is a fixed basis for the radical of α;

(2) the radical subgroup is a totally geodesic submanifold of G, i.e., it is formed by all geodesics issuing from the unity of G and tangent to $\mathrm{rad}\, \alpha$;

(3) the restriction of ∇ to D is the Levi-Civita connection of the metric $d\alpha_\Phi$ on the subgroup H generated by D.

PROOF. (1) Using Proposition 3.4 and arguing as in the proof of Theorem 2.5, we obtain (1).

(2) Applying the invariant definition of the Levi-Civita connection, we infer that

$$g(\nabla_X X, Y) = (1/2)(g(X, [X, Y]) - g(X, [Y, X]) + g(Y, [X, X])) = g(X, [X, Y])$$

for every X in $\operatorname{rad}\alpha = \mathfrak{r}$ and every Y in $\mathfrak{g}$. Since the metric g is Ad_R-invariant, the operator ad_X is skew-symmetric (see [9] for details). We have

$$g(X, [X, Y]) = -g([X, X], Y) = 0,$$

i.e., $\nabla_X X = 0$. Consequently, each curve of the form $\exp(X(t))$, where $X \in \mathfrak{r}$, is a geodesic with origin at the unity of G.

(3) Using the orthogonality of D and $\mathfrak{r}$ and the skew symmetry of ad_Z, we infer that X and Y in D and Z in $\mathfrak{r}$

$$g(\nabla_X Y, Z) = (1/2)(g(X, [Y, Z]) + g(Y, [X, Z]) + g(Z, [X, Y])) = (1/2)g(Z, [X, Y]) = 0,$$

i.e., $\nabla_X Y \in D$ for all X and Y in D.

Remark 3.6. If the affinor metric structure $g = (\alpha, \Phi, \beta)$ is Ad_H-invariant then H is the connected subgroup generated by D then the restriction of ∇ to $\mathfrak{r}$ is the Levi-Civita connection of β.

§ 4. K-Affinor Metric Structures

Let $G = (\alpha, \Phi, \beta)$ be an affinor metric structure on the Lie group G. By the Riesz Theorem about a linear functional (for example, see [10]), there exists a unique vector field ξ such that $\alpha(X) = g(\xi, X)$ for all X in TG. This vector field is called the *characteristic vector field* of the affinor metric structure. If the affinor metric structure g is left-invariant then its characteristic vector field ξ is also left-invariant, i.e., lies in $\mathfrak{g}$.

Definition 4.1. An affinor metric structure g is called K-*affinor* if its characteristic vector field ξ generates a one-parameter group of isometries of the metric g.

For a left-invariant affinor metric structure, this definition is equivalent to the assertion that the metric g is Ad_H-invariant, where H is the one-dimensional subgroup generated by the characteristic vector field.

The simplest example of a K-affinor metric structure is given by the affinor metric structure $g = (\alpha, \Phi, \alpha \otimes \alpha)$, where α is a contact structure on G with the characteristic vector field ξ which generates the one-dimensional radical of α and is a unit-length Killing vector field. These metric structures are called K-*contact* and studied in [1].

Proposition 4.2. *A left-invariant affinor metric structure g with characteristic vector field ξ is* K-*affinor if and only if the operator* ad_ξ *is skew-symmetric with respect to g.*

Proof. Let $h(t)$ be a one-parameter subgroup generated by the characteristic vector field ξ. Then

$$\frac{d}{dt} g(\mathrm{Ad}_{h(t)} X, \mathrm{Ad}_{h(t)} Y) = g([\xi, \mathrm{Ad}_{h(t)} X], \mathrm{Ad}_{h(t)} Y) + g(\mathrm{Ad}_{h(t)} X, [\xi, \mathrm{Ad}_{h(t)} Y])$$

for all X and Y in $\mathfrak{g}$. If g is $\mathrm{Ad}_{h(t)}$-invariant then, for $t = 0$, we have $g(\mathrm{ad}_\xi X, Y) + g(X, \mathrm{ad}_\xi Y) = 0$, and, conversely, if ad_ξ is skew-symmetric then

$$\frac{d}{dt} g(\mathrm{Ad}_{h(t)} X, \mathrm{Ad}_{h(t)} Y) = g([\xi, \mathrm{Ad}_{h(t)} X], \mathrm{Ad}_{h(t)} Y) + g(\mathrm{Ad}_{h(t)} X, [\xi, \mathrm{Ad}_{h(t)} Y]) = 0,$$

whence $g(\mathrm{Ad}_{h(t)} X, \mathrm{Ad}_{h(t)} Y) = g(X, Y)$ for all t.

As in the proof of Proposition 4.2, we may demonstrate that an affinor metric structure g is bi-invariant if and only if the operator ad_X is skew-symmetric with respect to g for every X in $\mathfrak{g}$. A detailed exposition may be found in [9].

Theorem 4.3. *Every connected Lie group of dimension ≥ 3 admits no bi-invariant metric structures with nontrivial radical.*

Proof. Suppose that a group G admits a bi-invariant affinor metric structure $g = (\alpha, \Phi, \beta)$ with nontrivial radical. Let R be the radical subgroup of α and let ξ be the characteristic vector field of the metric structure g. The subgroup R is a proper subgroup of G and coincides with the connected component of the unity of the isotropy subgroup of the adjoint action of G at α. Given h in G and X in $\mathfrak{g}$, we have

$$\mathrm{Ad}_h^* \alpha(X) = \alpha(\mathrm{Ad}_h X) = g(\mathrm{Ad}_h \xi, \mathrm{Ad}_h X) = g(\xi, X) = \alpha(X).$$

Thus, the isotropy subgroup coincides with the entire group G and is connected. Hence, $R = G$, which contradicts to the fact that the radical subgroup is a proper subgroup.

Proposition 4.4. *Suppose that $g = (\alpha, \Phi, \beta)$ is a left-invariant* K*-affinor metric structure on a Lie group G, ξ is its characteristic vector field, and ∇ is its Levi-Civita connection. Then*

(1) *$\xi \in \mathrm{rad}\,\alpha$;*
(2) *$\nabla_\xi X = -\Phi X$ and $\nabla_X \xi = -\Phi X - \mathrm{ad}_\xi X$ for every X in $\mathfrak{g}$;*
(3) *$g(\nabla_X Y, \xi) = g(X, \Phi Y)$ for all X and Y in $\mathfrak{g}$;*
(4) *$\mathrm{ad}_\xi = -2\Phi$.*

Proof. (1) By Proposition 4.2 we infer that, for every X in $\mathfrak{g}$,

$$d\alpha(\xi, X) = -(1/2)\alpha([\xi, X]) = -(1/2)g(\xi, [\xi, x]) = (1/2)g([\xi, \xi], X) = 0,$$

i.e., $\xi \in \mathrm{rad}\,\alpha$.

(2) Applying the invariant definition of the Levi-Civita connection and Proposition 4.2, we find that, for all X and Y in $\mathfrak{g}$,

$$\begin{aligned} g(\nabla_\xi X, Y) &= (1/2)(\xi, [X, Y]) + g(X, [\xi, Y]) + g([\xi, X], Y) = (1/2)g(\xi, [X, Y]) \\ &= (1/2)\alpha([X, Y]) = -d\alpha(X, Y) = -g(\Phi X, Y), \end{aligned}$$

i.e., $\nabla_\xi X = -\Phi X$, the equality $\nabla_X \xi = -\Phi X - \mathrm{ad}_\xi X$ follows from the condition that the torsion of ∇ is 0.

(3) Given X and Y in $\mathfrak{g}$, we have

$$\begin{aligned} g(\nabla_X Y, \xi) &= (1/2)(-g(X, [\xi, Y]) - g([\xi, X], Y) + g(\xi, [X, Y])) = (1/2)g(\xi, [X, Y]) \\ &= (1/2)\alpha([X, Y]) = -d\alpha(X, Y) = g(X, \Phi Y). \end{aligned}$$

(4) Using (2) and (3), for all X and Y in $\mathfrak{g}$ we infer

$$\begin{aligned} g(\mathrm{ad}_\xi X, Y) &= -g(\Phi X, Y) - g(\nabla_X \xi, Y) = -g(\Phi X, Y) + g(\xi, \nabla_X Y) \\ &= -g(\Phi X, Y) + g(X, \Phi Y) = -2g(\Phi X, Y), \end{aligned}$$

i.e., $\mathrm{ad}_\xi X = -2\Phi X$.

Proposition 4.5. *If a left-invariant vector field ξ lies in the center of the Lie algebra $\mathfrak{g}$ of a Lie group G then G admits no* K*-affinor metric structures with characteristic vector field ξ.*

Proof. Suppose that the group G admits a K-affinor metric structure g for which ξ is a characteristic vector field. By Proposition 4.4, we have $\Phi = -(1/2)\,\mathrm{ad}_\xi = 0$, which contradicts Definition 3.1.

Denote by E the kernel of α. The codimension of the distribution E is equal to 1, and the Lie algebra of G splits into the orthogonal sum of E and the straight line generated by the characteristic vector field ξ.

Theorem 4.6. *The sectional curvature k of a* K*-affinor metric structure $g = (\alpha, \Phi, \beta)$ on a Lie group G with characteristic vector field ξ is equal to* 1 *in every two-dimensional direction contained ξ.*

PROOF. Without loss of generality, we may prove the equality $k(\xi, X) = 1$ by assuming that the vector fields ξ and X have unit length and X lies in E. Since ξ lies in $\operatorname{rad}\alpha$, by Theorem 3.5 $\nabla_\xi \xi = 0$. Using Proposition 4.4, we infer

$$k(\xi, X) = g(\nabla_{[\xi,X]}\xi, X) + g(\nabla_\xi \nabla_X \xi, X) - g(\nabla_X \nabla_\xi \xi, X) = -g(\Phi \circ \operatorname{ad}_\xi X, X)$$
$$-g(\operatorname{ad}_\xi^2 X, X) - g(\Phi \nabla_X X, X) = 2g(X,X) + g(\Phi^2 X, X) = g(X,X) = 1.$$

Corollary 4.7. *The Ricci curvature of a* K*-affinor metric structure g on a Lie group G of dimension n in the direction of its characteristic vector field ξ is equal to $n-1$.*

PROOF. Fix an orthonormal basis $E_1, \dots, E_{n-1}$ for E. By Theorem 4.6, $k(E_i, \xi) = 1$ for all $i = 1, 2, \dots, n-1$. Then

$$\operatorname{Ric}(\xi) = \sum_{i=1}^{n-1} k(E_i, \xi) = n - 1.$$

Corollary 4.8. *A Lie group of dimension $n \geq 3$ admits no left-invariant* K*-affinor metric structures with negative sectional curvature. Every left-invariant* K*-affinor metric structure with positive definite sectional curvature has scalar curvature at least $n-1$.*

This is immediate from the fact that a K-affinor metric structure always has sectional curvatures equal to 1 in some directions and the scalar curvature of the Riemannian metric is equal to the sums of the Ricci curvatures along all basis directions.

§ 5. Normal Affinor Metric Structures

Suppose that $g = (\alpha, \Phi, \beta)$ is a left-invariant affinor metric structure on a Lie group G, while ξ is the characteristic vector field of g, $\mathfrak{r} = \operatorname{rad}\alpha$, and $D = \operatorname{rad}\beta$.

DEFINITION 5.1. A left-invariant metric structure g on a Lie group G is called *normal* if $\operatorname{ad}_{\Phi X} = \Phi \circ \operatorname{ad}_X$ for all X in the Lie group of G.

By Theorem 1.5, the dimension of D is always even. If the distribution D is the Lie subalgebra of the Lie algebra G and H is the connected subgroup generated by D then the affinor Φ defines a left-invariant almost complex structure on H. In [5, Chapter 9] it is proved that every almost complex structure J satisfying the condition $J \circ \operatorname{ad}_X = \operatorname{ad}_{JX}$ is integrable. Thus, the affinor Φ of the normal affine metric structure g defines a complex structure on the subgroup H, and the restriction of g to H is a Kähler metric on H.

We now prove that a normal affinor metric structure cannot be K-affinor.

Proposition 5.2. *The sets of left-invariant normal and* K*-affinor metric structures on a Lie group G do not intersect.*

PROOF. Let g be a normal affinor metric structure with affinor Φ and characteristic vector field ξ. If the metric structure g is K-affinor then, by Proposition 4.4(4), $\Phi = -(1/2)\operatorname{ad}_\xi$. Given X in $\mathfrak{g}$, we get

$$\operatorname{ad}_{\Phi X} = -(1/2)\operatorname{ad}_{[\xi,X]} = (1/2)(\operatorname{ad}_X \circ \operatorname{ad}_\xi - \operatorname{ad}_\xi \circ \operatorname{ad}_X).$$

On the other hand,

$$\operatorname{ad}_{\Phi X} = \Phi \circ \operatorname{ad}_X = -(1/2)\operatorname{ad}_\xi \circ \operatorname{ad}_X .$$

Comparing the right-hand sides of the last two equalities, we have $\operatorname{ad}_X \circ \operatorname{ad}_\xi = 0$, and, in particular, $\operatorname{ad}_\xi^2 = 0$. Thus, $\Phi^2 = 0$, which contradicts Definition 3.1.

Now, let G be a Lie group of dimension $2n$ and let J be a left-invariant complex structure on G preserving a normal affinor metric structure $g = (\alpha, \Phi, \beta)$. The fundamental 2-form of the Hermitian metric g has the form

$$\Omega(X, Y) = d\alpha(X, \Phi \circ JY) + \beta(X, JY).$$

Since the complex structure J also preserves the form β and $d\Omega(X, Y) = d\beta(X, JY)$, the normal affinor metric structure is Kähler if and only if the fundamental 2-form of the metric of the radical is closed.

Since the restriction of the fundamental 2-form of a Kähler normal metric structure g to the radical subgroup R is a left-invariant symplectic structure on R, we obtain the following result:

Proposition 5.3. *Suppose that α is a left-invariant nonclosed 1-form on a Lie group G, R is the radical subgroup of α, and J is a left-invariant complex structure on G. If G admits a left-invariant Kähler normal affinor metric structure associated with the 1-form α and the complex structure J then the radical subgroup R admits a left-invariant symplectic structure invariant under the complex structure J.*

Conversely, if the radical subgroup does not admit left-invariant symplectic structures invariant under the complex structure J then G admits no left-invariant Kähler normal affinor metric structures associated with the complex structure J.

If an almost complex structure J maps the subspaces $\mathfrak{r}$ and D into themselves then it may be identified with a pair of almost complex structures J_R and J_D, where J_R is the restriction of J to $\mathfrak{r}$ and J_D is the restriction of J to D. Such an almost complex structure is called *reducible*.

Let P be a left-invariant endomorphism of the Lie algebra of a Lie group G. Denote by A_P the left-invariant tensor field of the type $(2, 1)$ of the following form:

$$A_P(X, Y) = P[X, Y] - P[PX, PY] - [PX, Y] - [X, PY] \quad \text{for all } X \text{ and } Y \text{ in } \mathfrak{g},$$

and by $\mathbf{C}(\Phi)$, the set of all left-invariant endomorphisms of the Lie algebra of G anticommuting with the affinor Φ.

Theorem 5.4 (The Reduction Theorem). *Let $g = (\alpha, \Phi, \beta)$ be a left-invariant normal affinor metric structure on the Lie group G, let the distribution $D = \operatorname{rad}\beta$ be a subalgebra in $\mathfrak{g}$, let $J = (J_R, J_D)$ be a left-invariant reducible almost complex structure on G preserving $d\alpha$, and $A_P \equiv 0$ on D for every P in $\mathbf{C}(\Phi)$. Then J is integrable on G if and only if J_R is integrable on the radical subgroup R.*

Proof. In [11] it is proved that if J_0 is a fixed almost complex structure preserving the metric g then every almost complex structure J preserving the fundamental 2-form of g associated with the almost complex structure J_0 has the form

$$J = J_0 \circ (I + P) \circ (I - P)^{-1},$$

where I is the field of identity operators, P is the field of linear operators symmetric with respect to g and such that $J_0 \circ P = -P \circ J_0$ and $\det(I - P^2) \neq 0$. Since the metric structure g is normal, its affinor Φ is a complex structure on D. Choosing the affinor Φ as J_0, we infer that each almost complex structure J_D preserving $d\alpha$ has the form

$$J_D = \Phi \circ (I + P) \circ (I - P)^{-1} = (I - P) \circ \Phi \circ (I - P)^{-1},$$

where P is an endomorphism of the subalgebra D symmetric with respect to g and anticommuting with Φ.

Put $Q = I - P$. Since $A_P \equiv 0$ on D, for all X and Y in D we have

$$[QX, QY] = [X, Y] - [PX, Y] - [X, PY] + [PX, PY] = [X, Y] - P[X, Y] = Q[X, Y],$$

i.e., the linear operator Q is an automorphism of D and $J_D = Q \circ \Phi \circ Q^{-1}$. Since the almost complex structure J_D is isomorphic to the complex structure Φ, it is integrable on D.

Thus, the integrability of an almost complex structure J is reduced only to the integrability of J_R on the radical $\mathfrak{r}$.

Corollary 5.5. *Suppose that $g = (\alpha, \Phi, \beta)$ is a left-invariant normal affinor metric structure on a Lie group G, the distribution $D = \operatorname{rad}\beta$ is a subalgebra in $\mathfrak{g}$, J is a left-invariant reducible almost complex structure on G preserving $d\alpha$, and $A_P \equiv 0$ on D for every P in $\mathbf{C}(\Phi)$. If the radical subgroup R of α is commutative then J is integrable on G.*

Proof. In [5, Chapter 9] it is proved that an almost complex structure J is integrable if and only if

$$[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$$

for all vector fields X and Y. By the commutativity of R, the equality holds for all X and Y in $\mathfrak{r}$ and J is integrable by Proposition 5.3.

Let $g = (\alpha, \Phi, \beta)$ be a Hermitian normal affinor metric structure on a Lie group G. The metric structure g is called *locally conformally Kähler* if, for every $x \in G$, there exist a simply connected neighborhood U and a function f and a Kähler metric h defined in this neighborhood such that the restriction of g to U is equal to $\exp(-f)h$. It is proved in [12] that a Hermitian metric structure g with fundamental 2-form Ω is a locally conformally Kähler if and only if G admits a closed differential 1-form η such that $d\Omega = \eta \wedge \Omega$. If J is a reducible complex structure preserving the metric structure g then, by the equality $d\Omega(X,Y) = d\beta(X, JY)$, we infer that a normal affinor metric structure is locally conformally Kähler if and only if there is a closed 1-form η on G such that $d\beta_J = \eta \wedge \Omega$, where $d\beta_J(X,Y) = \beta(X, JY)$ for all X and Y in $\mathfrak{g}$. Obviously, η cannot coincide with α.

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