ISSN 0037–4466, Siberian Mathematical Journal, 2019, Vol. 60, No. 6, pp. 1022–1031. © *Pleiades Publishing, Inc., 2019.*
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SUBTWISTOR STRUCTURES AND SUBTWISTOR BUNDLE

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Abstract: We introduce the notions of subtwistor structure and subtwistor bundle. Under consideration is some particular case of subtwistor structures, sub-Kähler structures. The subtwistor bundle for the four-dimensional sphere is described. We also provide some examples of the manifolds that admit or do not admit any subtwistor structure. For a real manifold of arbitrary dimension, we give conditions for the existence of a sub-Kähler structure, which implies the existence of sub-Kähler submanifolds.

DOI: 10.1134/S0037446619060107

Keywords: subtwistor structure, sub-Kähler structure, subtwistor bundle, radical of an exterior form

§ 1. Introduction

The theory of Kähler and complex manifolds contains the notions of twistor bundle and twistor structure. This article is devoted to the generalization of the notion of twistor bundle, the subtwistor bundle. In [1], we introduced the notion of subtwistor bundle as a generalization of the twistor bundle to the case of a degenerate skew-symmetric 2-form on a manifold of arbitrary dimension. For constructing the subtwistor bundle on a manifold M of arbitrary dimension, we have to define a skew-symmetric 2-form Ω at each point of M, the set of all affinors associated with Ω at this point, and the set of all possible interior products in the vector space $\operatorname{rad}\Omega_x$ which is the radical (in another terminology, the kernel) of the 2-form Ω at a point x. Here the 2-form Ω can be degenerate, i.e. have nonzero radical. In contrast to the classical twistor bundle, the subtwistor bundle can be defined for a manifold of any even or odd dimension. In the present article, we provide some examples of the manifolds that admit or do not admit any subtwistor structure and describe the subtwistor bundles of a few manifolds.

Interest in the study of degenerate skew-symmetric 2-forms is due to the problems of finding symplectic and Kähler submanifolds of manifolds of arbitrary dimension. In [2], homogeneous spaces with invariant degenerate skew-symmetric closed 2-form are studied, and the articles [1, 3] deal with the sub-Kähler structures that enable us to obtain Kähler submanifolds. Moreover, subtwistor structures arise in various physical applications of Riemannian geometry, in particular, in string theory, analytical mechanics, and the theory of Calabi–Yau spaces.

To define a subtwistor structure on a manifold M, we must endow M with a subtwistor structure, a regular skew-symmetric 2-form Ω with nonzero radical $\operatorname{rad}\Omega$, a complementary distribution of tangent spaces D to $\operatorname{rad}\Omega$ on which the-2-form Ω is nondegenerate, an affinor Φ that is a generalization of the notion of an almost complex structure associated with Ω, and an interior product on the distribution $\operatorname{rad}\Omega$, called a radical metric, which enables us to obtain a global Riemannian metric on M. Subtwistor structures with closed 2-form Ω make it possible to introduce the notion of sub-Kähler structure (see [1]), and subtwistor structures with exact 2-form Ω make it possible to obtain the affinor metric structures (see [4, 5]) that generalize contact metric structures for manifolds of any dimension and Lie algebroids. In this article, we show how, using subtwistor and sub-Kähler structures, we can obtain Kähler submanifolds of a manifold of arbitrary dimension as well as the subtwistor bundle for the manifolds that do not admit a symplectic or Kähler structure.

Section 2 contains the necessary information and properties for the subtwistor structures introduced in [1]. Section 3 deals with a particular case of subtwistor structures, sub-Kähler structures, and give the conditions under which a subtwistor structure defines a sub-Kähler structure. In Section 4, we introduce the notion of subtwistor bundle and show how it is related to the existence of a subtwistor

Original article submitted December 26, 2018; revised March 25, 2019; accepted May 15, 2019.

structure on a manifold. In Section 5, we describe the subtwistor bundle of the four-dimensional sphere. Section 6 is devoted to the study of invariant subtwistor structures and contains homogeneous examples of subtwistor structures. We use the results and notions that had been described in [1, 3].

§ 2. Subtwistor Structures

Let M be a smooth real manifold of dimension $n \geq 3$, let Ω be a bilinear form on M, and let X be a smooth vector field on M. Denote by $\mathrm{I}_X \Omega$ the interior product of X and Ω, whose result is the 1-form $\mathrm{I}_X \Omega$ such that $\mathrm{I}_X \Omega(Y) = \Omega(X, Y)$ for every vector field Y on M.

Definition 2.1. The *radical of* Ω *on* M *at* x is the tangent subspace $\operatorname{rad}\Omega_x = \{v \in T_xM : \mathrm{I}_v \Omega_x = 0\}$. The form Ω is called *regular* if the distribution of radicals $\operatorname{rad}\Omega$ has constant rank on M.

We will denote by $\operatorname{rad}\Omega$ the distribution of radicals of a bilinear form Ω on a manifold M. Sometimes, the radical of Ω is called the *kernel* of Ω. Obviously, Ω is nondegenerate if and only if $\operatorname{rad}\Omega = \{0\}$.

Theorem 2.2. *Let M be a smooth manifold of dimension $n \geq 3$, let Ω be a skew-symmetric regular 2-form on M, and let r be the rank of* $\operatorname{rad}\Omega$. *Then*

(1) *if n is even then so is r and $0 \leq r \leq n-2$;*

(2) *if n is odd then so is r and $1 \leq r \leq n-2$;*

(3) *if $d\Omega = 0$ then* $\operatorname{rad}\Omega$ *is a holonomic distribution on M.*

Proof. Demonstration of items (1) and (2) can be found in [4], while the proof of item (3), in [1]. Let D be a distribution of tangent subspaces on M complementary to $\operatorname{rad}\Omega$ such that the restriction of Ω to every fiber of D is nondegenerate. Such a distribution is called a *work bundle* for Ω. Theorem 2.2 implies that a work bundle D has even rank for a manifold of any dimension. This fact enables us to introduce the important notion of affinor associated with a regular skew-symmetric 2-form.

Definition 2.3. Let Ω be a regular skew-symmetric 2-form on a manifold M and let D be a work bundle for Ω. By an *affinor associated with* Ω we mean a continuous field Φ of endomorphisms of tangent subspaces on M satisfying the conditions:

(1) $\ker\Phi = \operatorname{rad}\Omega$;

(2) $\Phi^2|_D = -\mathrm{id}$, where id is the field of identity operators on M;

(3) $\Omega \circ \Phi = \Omega$;

(4) $\Omega(X, \Phi X) \geq 0$ for every vector field $X \in C^1(TM)$.

The following important properties of an affinor are immediate from Definition 2.3.

Proposition 2.4. *Let Φ be an affinor associated with a regular skew-symmetric 2-form Ω on M and let D be a work bundle for Ω.*

(1) *The restriction of Φ to D is a complex structure in the fibers of D preserving Ω.*

(2) *Φ is an affinor associated with $\lambda\Omega$ for every continuous function λ on M such that $\lambda(x) > 0$ for all $x \in M$.*

(3) *$-\Phi$ is an affinor associated with $\mu\Omega$ for every continuous function μ on M such that $\mu(x) < 0$ for all $x \in M$.*

Let Ω be a regular skew-symmetric 2-form on a manifold M with work bundle D and let Φ be an affinor associated with Ω. Denote by Ω_Φ the symmetric 2-form on M such that $\Omega_\Phi(X, Y) = \Omega(X, \Phi Y)$ for all vector fields $X, Y \in C^1(TM)$.

Definition 2.3 implies that $\operatorname{rad}\Omega_\Phi = \operatorname{rad}\Omega$ and the restriction of Ω_Φ to D is an interior product in the fibers of D. A *radical metric* of a bilinear regular form Ω on M is a symmetric bilinear form β on M such that $\operatorname{rad}\beta = D$, where D is a work bundle for Ω and the restriction of β to $\operatorname{rad}\Omega$ is an interior product in the fibers of $\operatorname{rad}\Omega$. Endow M with the Riemannian metric $g = \Omega_\Phi + \beta$. To construct this, we need some collection of objects (Ω, D, Φ, β) where Ω is a regular skew-symmetric 2-form on M and D is a work bundle for Ω, while Φ is an affinor associated with Ω, and β is a radical metric of Ω. Such a collection of objects is called a *subtwistor structure* on M. Obviously, if $\operatorname{rad}\Omega = \{0\}$ then $D = TM$,

$\beta = 0$, and (Ω, Φ) is a twistor structure on M. The alternative definition of subtwistor structure as well as the properties and examples of subtwistor structures can be found in [1].

For studying the question of existence of a subtwistor structure on a manifold, we will need some topological conditions that are connected with characteristic classes. Let E be a vector bundle over a smooth manifold M, let $e(E)$ be the Euler class of E, and let $w_1(E)$ be the first Stiefel–Whitney class of E. If M admits an everywhere nonzero global section then $e(E) = 0$. If the fibers of a vector bundle E are endowed with orientation depending continuously on a point $x \in M$ and M is orientable then $w_1(E) = 0$ (see [6]).

Let (Ω, D, Φ, β) be a subtwistor structure on a manifold M and let $\Lambda^2(M)$ be the bundle of skew-symmetric 2-forms on M. Since Ω is a regular 2-form on M; therefore, $\Omega_x \neq 0$ for all $x \in M$ and Ω is a global section of $\Lambda^2(M)$. Since each complex structure in the vector space D_x defines orientation in D_x and Φ is a complex structure in the fibers of D, we obtain

Proposition 2.5. *If an orientable manifold M admits a subtwistor structure with work bundle D then $e(\Lambda^2(M)) = 0$ and $w_1(D) = 0$.*

If M is a compact orientable manifold without boundary and $\chi(M)$ is the Euler characteristic of M then $\chi(M) = \int_M e(M)$, where $e(M)$ is the Euler class of the tangent bundle TM (see [6]). This yields

Corollary 2.6. *If the total space of the bundle $\Lambda^2(M)$ over a manifold M is a compact orientable manifold without boundary and $\chi(\Lambda^2(M)) \neq 0$ then M does not admit subtwistor structures.*

Let us discuss how subtwistor structures are connected with the problem of obtaining Kähler submanifolds of an arbitrary manifold.

§ 3. Sub-Kähler Structures

Let M be a real smooth manifold of dimension ≥ 3 and let (Ω, D, Φ, β) be a subtwistor structure on M. We will refer to Ω as the *fundamental 2-form of the subtwistor structure*. In general, the work bundle D is not a holonomic distribution on M. If $d\Omega = 0$ and D is a holonomic distribution on M then in M there is a submanifold $Q : D|_Q = TQ$ and the restriction of the affinor Φ to Q is an almost complex structure on Q. Putting $\Omega_\Phi(X, Y) = \Omega(X, \Phi Y)$, $X, Y \in C^1(TM)$, we infer that $(\Omega, \Phi, \Omega_\Phi)$ is an almost Kähler structure on Q.

Definition 3.1. A *sub-Kähler structure on a manifold M of dimension* ≥ 3 is a collection of objects $(Q, \Omega, D, \Phi, \beta)$, where Ω is a closed regular skew-symmetric form on M and D is a work bundle for Ω, while Φ is an affinor associated with Ω, β is a radical metric of Ω, and Q is a submanifold in M such that $TQ = D|_Q$ and the restriction of Φ to Q is a complex structure on Q.

It is obvious from the definition that the existence of a sub-Kähler structure on a manifold M of arbitrary dimension implies the existence of Kähler submanifolds of M. It remains to find out the conditions for a subtwistor structure with closed fundamental 2-form to induce a sub-Kähler structure on M.

Definition 3.2. The *torsion of a subtwistor structure* (Ω, D, Φ, β) on a manifold M is the continuous tensor field N of type $(2, 1)$ defined at a pair of vector fields $X, Y \in C^1(TM)$ as follows:

$$N(X, Y) = [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] + \Phi^2[X, Y],$$

where $[X, Y]$ is the Lie bracket of X and Y.

Let N be the torsion tensor of a subtwistor structure (Ω, D, Φ, β) on M and let $d\Omega = 0$. It was proved in [1] that the condition $N = 0$ implies that the work bundle D is a holonomic distribution on M and, for any integral submanifold $Q : TQ = D|_Q$, the restriction of Φ to Q is a complex structure on Q. Thus, we obtain

Proposition 3.3. *Let M be a real manifold of dimension $n \geq 3$ and let (Ω, D, Φ, β) be a subtwistor structure on M with radical of rank $r \geq 1$, closed fundamental 2-form Ω, and zero torsion tensor. Then the work bundle D is a holonomic distribution on M whose every integral submanifold $Q : TQ = D|_Q$ is a Kähler submanifold of complex dimension $\frac{n-r}{2}$ and $(Q, \Omega, D, \Phi, \beta)$ is a sub-Kähler structure on M.*

Since every almost complex structure on a two-dimensional real manifold is integrable (see [8, Chapter 9]), we obtain

Proposition 3.4. *Let M be a real manifold of dimension $n \geq 3$. Each subtwistor structure (Ω, D, Φ, β) on M with closed fundamental 2-form Ω, radical of maximal rank $n-2$, and involutive work bundle D, generates the sub-Kähler structure $(Q, \Omega, D, \Phi, \beta)$ on M, where Q is the maximal integral submanifold for D.*

The simplest example of a manifold with sub-Kähler structure is given by the direct product of a Kähler manifold and a Riemannian manifold. Some nontrivial class of examples can be obtained by the so-called normal subtwistor structure. A subtwistor structure (Ω, D, Φ, β) is called *normal* if $[\Phi X, Y] = \Phi[X, Y]$ for all $X \in C^1(D)$ and $Y \in C^1(TM)$.

Proposition 3.5. *Suppose that M is a real manifold of dimension $v \geq 3$, (Ω, D, Φ, β) is a normal subtwistor structure on M, and $d\Omega = 0$. Then D is a holonomic distribution on M, every integral submanifold $Q : TQ = D|_Q$ is a Kähler submanifold of M, and the set of all smooth sections of D is an ideal in the space of vector fields on M.*

Proof. Let N be the curvature tensor of (Ω, D, Φ, β). By Definition 3.2,

$$\begin{aligned} N(X,Y) &= [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] + \Phi^2[X,Y] \\ &= -\Phi^2[X,Y] + \Phi^2[X,Y] - \Phi^2[X,Y] + \Phi^2[X,Y] = 0 \end{aligned}$$

for all $X, Y \in C^1(D)$. Using (1) of Definition 2.3 and given $X \in C^1(D)$ and $Y \in C^1(\operatorname{rad}\Omega)$, we have

$$N(X,Y) = -\Phi[\Phi X, Y] + \Phi^2[X,Y] = 0.$$

Since $N|_{\operatorname{rad}\Omega} = 0$, we infer finally that $N = 0$ on M. Proposition 3.3 implies that D is a holonomic bundle on M and any integral submanifold Q: $TQ = D|_Q$ is a Kähler submanifold in M.

Since D is a holonomic distribution on M, the Frobenius Theorem implies that D is an involutive distribution on M. Definition 2.3 implies that $\Phi X \in C^1(D)$ for all $X \in C^1(TM)$. Given $X \in C^1(D)$ and $Y \in C^1(\operatorname{rad}\Omega)$, we have $[\Phi X, Y] = \Phi[X,Y] \in C^1(D)$. Since Φ is a linear automorphism of the fibers of D and $C^1(TM) = C^1(D) \oplus C^1(\operatorname{rad}\Omega)$, we conclude that $C^1(D)$ is an ideal of $C^1(TM)$.

Remark 3.6. Proposition 3.5 and the Frobenius Theorem imply that, for a normal subtwistor structure (Ω, D, Φ, β), there exist integral submanifolds $Q : TQ = D|_Q$ and $R : TR = \operatorname{rad}\Omega|_R$ but M is locally isometric to $Q \times R$ only if $[D, \operatorname{rad}\Omega] = 0$.

§ 4. The Subtwistor Bundle

Let V be a vector space of dimension ≥ 3 over the field $\mathbb{R}$. A *subtwistor structure* with radical of dimension r in V is a collection of objects (Ω, D, Φ, β), where Ω is a skew-symmetric bilinear form on V with radical of dimension r and D is an even-dimensional subspace in V such that the restriction of the form Ω to D is nondegenerate, while Φ is an affinor associated with the form Ω as in Definition 2.3, β is a symmetric bilinear form on V such that $\operatorname{rad}\beta = D$ and the restriction of β to $\operatorname{rad}\Omega$ is an interior product on $\operatorname{rad}\Omega$. By a bundle over a manifold M we mean a manifold P with projection $\pi : P \to M$, where π is a continuous surjective mapping onto M. Proposition 2.4 implies that, for every constant $\lambda > 0$, the 2-forms Ω and $\lambda\Omega$ on V have the same set of associated affinors. We will assume that the subtwistor structures $(\Omega_1, D_1, \Phi_1, \beta_1)$ and $(\Omega_2, D_2, \Phi_2, \beta_2)$ are equivalent if $\Omega_2 = \lambda\Omega_1$, $\lambda > 0$, $D_2 = D_1$, $\Phi_2 = \Phi_1$, and $\beta_2 = \beta_1$.

Definition 4.1. The *subtwistor bundle with radical of rank r over a manifold M* is the bundle P with projection $\pi : P \to M$ such that $\pi^{-1}(x)$ is the set of cosets of the subtwistor structures with radical of dimension r in the tangent space T_xM for all $x \in M$.

A subtwistor structure with radical of rank r on a manifold M is a global section of the subtwistor bundle over M with radical of rank r. For $r = 0$, each affinor at $x \in M$ is an orthogonal almost complex structure at x. It follows that the classical twistor bundle is a subbundle of the subtwistor bundle with radical of rank 0.

Proposition 4.2. *Let M be a paracompact manifold of dimension $n \geq 3$. Every fiber of the subtwistor bundle with radical of rank $r = n - 2k$ over M is isomorphic to $\mathrm{gr}^{2k} \times \mathrm{SO}(2k)/\,\mathrm{U}(k) \times \mathrm{SM}(r)$, where gr^{2k} is the $2k$-Grassmannian in $\mathbb{R}^m$, while $\mathrm{SO}(2k)$ is the group of all orthogonal $2k \times 2k$-matrices with determinant 1, $\mathrm{U}(k)$ is the group of all Hermitian $k \times k$-matrices, and $\mathrm{SM}(r)$ is the space of nondegenerate positive-defined symmetric $r \times r$-matrices.*

Proof. A paracompact manifold always admits a Riemannian metric g (see [7]). If (Ω, D, Φ, β) is a subtwistor structure with radical of rank r in the vector space $V = T_xM,\ x \in M$; then the metric g defines the interior product $(\cdot,\cdot)$ in V. Choose a $2k$-frame u in V, which is an isomorphism $\mathbb{R}^{2k} \to D$, where D is the tangent space in V generated by u. Let $\mathrm{L}(n, 2k, \mathbb{R})$ be the set of all real $n \times 2k$-matrices of rank $2k$ and let $\mathrm{GL}(2k, \mathbb{R})$ be the set of all nondegenerate real $2k \times 2k$-matrices embedded in $\mathrm{L}(n, 2k, \mathbb{R})$. We will assume that $a \in \mathrm{L}(n, 2k, \mathbb{R})$ acts at the frame u from the right. Then the set of all tangent subspaces of dimension $2k$ in V is isomorphic to $\mathrm{L}(n, 2k, \mathbb{R})/\,\mathrm{GL}(2k, \mathbb{R}) \cong \mathrm{gr}^{2k}$.

Definition 2.3 implies that the affinor Φ is identified with an orthogonal complex structure in a fiber of the work bundle D. The frame u defines an isomorphism between the affinors associated with Ω and the orthogonal complex structures in $\mathbb{R}^{2k}$; i.e., $J_D = u \circ J \circ u^{-1}$, where J_D is a complex structure in D and J is a complex structure in $\mathbb{R}^{2k}$. The set of all orthogonal complex structures on $\mathbb{R}^{2k}$ is isomorphic to $\mathrm{SO}(2k)/\,\mathrm{U}(k)$ (see [8, Chapter 9]). Moreover, since every radical metric in V can be obtained from the interior product induced by the metric g with the use of a nondegenerate positive-defined $r \times r$-matrix (see [7]), the set of all radical metrics in V is isomorphic to $\mathrm{SM}(r)$.

Show that a vector subspace D of dimension $2k$ in V and a complex structure J in D orthogonal with respect to g uniquely define a coset of skew-symmetric 2-forms with radical of rank $r = n - 2k$. Denote by R the orthogonal complement to D in V with respect to g and let Φ stand for the endomorphism of V such that $\Phi X = JX$ if $X \in D$ and $\Phi X = 0$ if $X \in R$. Define in V the bilinear form $\Omega(X, Y) = g(\Phi X, Y)$, $X, Y \in V$. The properties of the orthogonal complex structure J and the construction of Φ imply that Ω is a skew-symmetric 2-form with radical R and work subspace D. Conversely, for every skew-symmetric 2-form with radical of rank r, there is an equivalent 2-form Ω in V whose matrix in a basis of the subspace D orthonormal with respect to g is the matrix of a complex structure in D orthogonal with respect to g. Here D is chosen uniquely as the orthogonal complement to $\operatorname{rad}\Omega$ with respect to g. Thus, we obtain a one-to-one correspondence between the cosets of skew-symmetric 2-forms with radical of rank r in V and the pairs (D, J), where D is a subspace of dimension $2k = n - r$ in V, and J is a complex structure in D orthogonal with respect to g. We conclude finally that the fiber of the subtwistor bundle at a point x is isomorphic to $\mathrm{gr}^{2k} \times \mathrm{SO}(2k)/\,\mathrm{U}(k) \times \mathrm{SM}(r)$.

Remark 4.3. The proof of Proposition 4.2 implies that the dimension of the fiber of the subtwistor bundle with radical of rank r over a paracompact manifold M of dimension $n \geq 3$ is equal to $\frac{(n-r)^2}{4} + \frac{r}{2}\,(2n - r + 1)$.

A subtwistor bundle with radical of rank r over a manifold M of dimension $n \geq 3$ isomorphic to $M \times \mathrm{gr}^{2k} \times \mathrm{SO}(2k)/\,\mathrm{U}(k) \times \mathrm{SM}(r)$, $2k = n - r$, is called *trivial*. For a trivial subtwistor bundle, each subtwistor structure in $\mathbb{R}^n$ induces a global section of the subtwistor bundle on M. Using Corollary 2.6, we obtain

Proposition 4.4. *Let $\Lambda^2(M, r)$ be the bundle of cosets of skew-symmetric 2-forms with radical r over a manifold M, where two 2-forms are assumed equivalent if they coincide up to multiplication by a positive real number, let $\Lambda^2(M, r)$ be an orientable compact manifold without boundary, and let $\chi(\Lambda^2(M, r))$ be the Euler characteristic of $\Lambda^2(M, r)$. If the subtwistor bundle with radical of rank r over M is trivial then $\chi(\Lambda^2(M, r)) = 0$.*

Proposition 4.5. *Let M be a paracompact manifold of dimension $n \geq 3$ and let P^r be the subtwistor bundle with radical of rank $r = n - 2k$ over M. If there is a global $2k$-frame ($2k$-coframe) on M then P^r is trivial.*

Proof. Since a paracompact manifold always admits a Riemannian metric and a Riemannian metric defines an isomorphism between $2k$-frames and $2k$-frames, it suffices to assume that a global $2k$-frame u is defined on M. At each $x \in M$, the frame u uniquely defines the $2k$-dimensional tangent space D_x and u is an isomorphism between D_x and $\mathbb{R}^{2k}$ (see [8]). As in the proof of Proposition 4.2, identify the set of all tangent subspaces of dimension $2k$ in T_xM with the space $\mathrm{L}(n, 2k, \mathbb{R})/\operatorname{GL}(n, \mathbb{R}) \cong \mathrm{gr}^{2k}$. Denote by $D_x a$ the tangent subspace in T_xM generated by the frame ua, $a \in \mathrm{L}(n, 2k, \mathbb{R})/\operatorname{GL}(n, \mathbb{R})$. Let J be a complex structure in $\mathbb{R}^{2k}$ and $J(ua) = ua \circ J \circ (ua)^{-1}$. Then $J(ua)$ is a complex structure in $D_x a$ and u induces an isomorphism between the set of all complex structures in $D_x a$ and $\mathrm{SO}(2k)/\mathrm{U}(k)$. The proof of Proposition 4.2 implies that the Riemannian metric g on M defines an isomorphism between the set of all radical metrics at the point x and $\mathrm{SM}(r)$. Since u and g depend continuously on x and are defined globally on M; therefore, P^r is trivial.

A manifold M is called *parallelizable* if M admits a global n-frame, where $n = \dim(M)$. This frame makes it possible to introduce a natural Riemannian metric on M. Now, Proposition 4.5 yields

Corollary 4.6. *Let M be a parallelizable manifold of dimension ≥ 3. Then the subtwistor bundle with radical of any possible rank over M is trivial.*

REMARK 4.7. If a manifold P is a trivial vector bundle of rank $m \geq 3$ then P admits the subtwistor structure induced by the subtwistor structure in $\mathbb{R}^m$. However, the subtwistor bundle over P can be nontrivial.

Obviously, on a manifold with trivial subtwistor bundle of rank r, there exists a subtwistor structure with radical of rank r as the standard section of the subtwistor bundle. Below we will consider the example of a manifold on which there are no subtwistor structures but the subtwistor bundle is nontrivial.

§ 5. The Subtwistor Bundle over the Four-Dimensional Sphere

Let S^4 be the four-dimensional sphere embedded in $\mathbb{R}^5$ which is a compact orientable manifold without boundary with the Riemannian metric induced from $\mathbb{R}^5$. It is known that the classical twistor bundle over S^4 is the complex projective space $\mathbb{C}P^3$ with fiber $\mathbb{C}P^1$ (see [9]). Here we describe the most general case of the subtwistor bundle with radical of any rank. Theorem 2.2 implies that the subtwistor bundle over S^4 can only have a radical of rank 0 or 2. The set of all radical metrics on a Riemannian manifold can be identified with the space of all nondegenerate positive-defined symmetric $r \times r$-matrices, where r is the rank of the radical. By Proposition 4.2, we have to describe the bundle $\mathrm{gr}^{2k}(S^4) \times \mathscr{J}(\mathrm{gr}^{2k}(S^4))$, where $\mathrm{gr}^{2k}(S^4)$ is the bundle of tangent subspaces of dimension $2k$ over S^4, while $\mathscr{J}(\mathrm{gr}^{2k}(S^4))$ is the bundle of orthogonal complex structures in the fibers of $\mathrm{gr}^{2k}(S^4)$, $2k = 4 - r$.

The complex space $\mathbb{C}^4$ can be identified with the quaternionic space $\mathbb{H}^2$ on assuming that $q_1 = z_1 + z_2 j$ and $q_2 = z_3 + z_4 j$ where $j = \sqrt{-1}$. Introduce the two functions on $\mathbb{H}^2 \setminus \{0\}$:

$$f(q_1, q_2) = 2\,\frac{q_1 \bar{q}_2}{|q_1|^2 + |q_2|^2}, \quad h(q_1, q_2) = \frac{|q_1|^2 - |q_2|^2}{|q_1|^2 + |q_2|^2}.$$

Note that $|f(q_1, q_2)|^2 + |h(q_1, q_2)|^2 = 1$. These functions are invariant under multiplying (q_1, q_2) by a nonzero quaternion. Introduce the projection $\pi : \mathbb{H}^2 \setminus \{0\} \to S^4$, $\pi(q_1, q_2) = (f(q_1, q_2), h(q_1, q_2)) \in \mathbb{R}^5$. Let the group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ act on $\mathbb{H}^2$ by homotheties. Then $\mathbb{C}P^3$ is $(\mathbb{H}^2 \setminus \{0\})/\mathbb{C}^*$. Since π is $\mathbb{C}^*$-invariant, π is the projection $\mathbb{C}P^3 \to S^4$. Each nonzero quaternion q generates a complex straight line in $\mathbb{C}^2$. If $q \neq 0$ then $\pi(qq_1, qq_2) = \pi(q_1, q_2)$. Hence, the fiber of $\mathbb{C}P^3 \to S^4$ is $(\mathbb{H} \setminus \{0\})/\mathbb{C}^* \cong \mathbb{C}P^1 \cong S^2$.

Suppose that $z = (q_1, q_2) \in \mathbb{C}P^3$ and $x = \pi(q_1, q_2) \in S^4$, while h is the Fubini–Study metric on $\mathbb{C}P^3$ (see [7]), and J_0 is the complex structure on $\mathbb{C}P^3$ induced by multiplication by the imaginary unity. Denote by D_z the orthogonal complement to $T_z\mathbb{C}P^1$ in $T_z\mathbb{C}P^3$ with respect to h. Then $T_z\mathbb{C}P^3 = D_z \oplus T_z\mathbb{C}P^1$. Since the complex structure J_0 is orthogonal and acts invariantly on $T_z\mathbb{C}P^1 \cong \mathbb{C}$, the restriction of J_0 to D_z is a complex structure in D_z. Denote the restriction of J_0 to D_z by I_z. This complex structure defines the complex structure $J_z = d\Pi \circ I_z d\pi^{-1}$ in the tangent space T_xS^4. Note that the complex structure J_z is orthogonal with respect to the Riemannian metric on S^4 induced by the metric h

on $\mathbb{C}P^3$ and preserves orientation in T_xS^4 since I_z preserves orientation in D_z. Similarly, the complex structure $-J_0$ defines the orthogonal complex structure $-J_z$ in T_xS^4 which changes orientation in T_xS^4. Thus, with each point $z = (q_1, q_2) \in \mathbb{C}P^3$, we can continuously associate a point $x = \pi(q_1, q_2) \in S^4$ and a pair of orthogonal complex structures in T_xS^4.

Let $\mathbb{H}P^1$ be the set of all quaternionic straight lines in $\mathbb{H}^2$ which is an orientable compact boundaryless manifold diffeomorphic to S^7/S^3. Since $\mathrm{gr}^4(S^4) \cong S^4$ and $S^4 \cong \mathbb{H}P^1$, we obtain

Proposition 5.1. *The subtwistor bundle with radical of rank* 0 *over the four-dimensional sphere is isomorphic to* $\mathbb{H}P^1 \times \mathbb{C}P^3 \times \mathbf{z}_2$.

Let $\Lambda^2(S^4, 0)$ be the bundle of cosets of skew-symmetric 2-forms with radical of rank 0 over S^4, like in Proposition 4.4, and let $\chi(M)$ be the Euler characteristic of M. Since $\chi(\mathbb{H}P^1) = 3$, $\chi(\mathbb{C}P^3) = 4$, and $\chi(\mathbf{z}_2) = 2$, we obtain

$$\chi(\Lambda^2(S^4, 0)) = \chi(\mathbb{H}P^1)\chi(\mathbb{C}P^3)\chi(\mathbf{z}_2) = 24.$$

From Proposition 4.4 we deduce that the subtwistor bundle with radical of rank 0 is nontrivial. Moreover, Corollary 2.6 yields

Corollary 5.2. *The four-dimensional sphere does not admit subtwistor structures with radical of rank* 0.

Remark 5.3. Since every almost complex structure on a Riemannian manifold M is orthogonal with respect to some Riemannian metric, and every orthogonal almost complex structure on M together with a Riemannian metric induces a subtwistor structure with radical of rank 0 on M (see § 2), Corollary 5.2 implies that the four-dimensional sphere does not admit almost complex structures.

Consider the embedding of $\mathbb{C}P^2$ into $\mathbb{C}P^3$ by assuming that $q_2 = z_2 \in \mathbb{C}$ in a pair $(q_1, q_2) \in \mathbb{C}P^3$. We have $(qq_1, qz_2) \in \mathbb{C}P^2$ only if $qz_2 \in \mathbb{C}$. This is possible only if q is a nonzero complex number. Note that $\Pi(-q_1 i, z_2) = \pi(q_1, iz_2)$, where $i = \sqrt{-1}$, and the points $(-q_1 i, z_2)$ and (q_1, iz_2) do not lie on the same complex straight line since $-q_1 i \neq -iq_1$. We infer that the fiber under the restriction of π to $\mathbb{C}P^2$ is $\mathbf{z}_2$. Each bundle with fiber $\mathbf{z}_2$ is a two-sheeted covering. It follows that $\mathbb{C}P^2$ covers S^4 in a two-sheeted way. Suppose that $z = (q_1, z_2) \in \mathbb{C}P^2$, while J_z is the complex structure in $T_z\mathbb{C}P^2$ induced by multiplication by the imaginary unit, and L_z is a complex straight line in $T_z\mathbb{C}P^2 \cong \mathbb{C}^2$. Since J_z is a complex structure orthogonal with respect to the Fubini–Study metric on $\mathbb{C}P^2$ and the complex straight line L_z is invariant under multiplication by the imaginary unity, the restriction of the complex structure J_z to L_z is an orthogonal complex structure in L_z preserving orientation in L_z. Since in $\mathbb{R}^2$ there is a unique orthogonal orientation-preserving complex structure and the restriction of the complex structure $-J_z$ to L_z is the only orthogonal orientation-reversing complex structure in L_z, we infer that every point $z \in \mathbb{C}P^2$ defines exactly two orthogonal complex structures in L_z. Since $d\pi L_z$ is a two-dimensional tangent subspace in T_xS^4, with $x = \pi(z) \in S^4$, and T_xS^4 admits only two orthogonal complex structures $\pm J_x = d\pi \circ \pm J_z \circ d\pi^{-1}$; we obtain

Proposition 5.4. *The subtwistor bundle with radical of rank* 2 *over the four-dimensional sphere is isomorphic to* $\mathbb{C}P^1 \times \mathbb{C}P^2 \times \mathrm{SM}(2)$.

Let $\Lambda^2(S^4, 2)$ be the bundle of cosets of skew-symmetric 2-forms with radical of rank 2 over S^4 as in Proposition 4.4. Propositions 5.4 and 4.2 imply that $\Lambda^2(S^4, 2) \cong \mathbb{C}P^1 \times \mathbb{C}P^2$. We have

$$\chi(\Lambda^2(S^4, 2)) = \chi(\mathbb{C}P^1)\chi(\mathbb{C}P^2) = 6.$$

From Proposition 4.4 we infer that the subtwistor bundle with radical of rank 2 is nontrivial. Moreover, Corollary 2.6 yields

Corollary 5.5. *The four-dimensional sphere does not admit subtwistor structures with radical of rank* 2.

Note that we described all possible types of the subtwistor bundle over S^4. Observe that S^4 is a homogeneous space without any subtwistor structures. Further we will consider conditions under which there can exist some invariant subtwistor structures on a homogeneous space.

§ 6. Invariant Subtwistor Structures on Homogeneous Spaces

Let $M = G/H$ be a homogeneous space of dimension ≥ 3, where G is a Lie group acting transitively and effectively on M, while H is the isotropy subgroup of the origin point $o \in M$. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G and let $\mathfrak{h}$ be the isotropy subalgebra of $\mathfrak{g}$. In $\mathfrak{g}$, we can choose the complementary subspace $\mathfrak{m} : \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. The subspace $\mathfrak{m}$ is isomorphic to the tangent space T_oM. Let π be the projection $G \to M$. Then $\tau = d\pi_e$, with e the unity of G, is an isomorphism $\mathfrak{m} \to T_oM$. A subtwistor structure (Ω, D, Φ, β) on a homogeneous space M is called *G-invariant* if D is a G-invariant distribution of tangent subspaces on M, and for every $g \in G$ we have $\Omega_o = \Omega_x \circ dg$, $\beta_o = \beta_x \circ dg$, and $\Phi_x \circ dg = dg \circ \Phi_o$; here $x = g(o)$, while dg is the differential of $g : M \to M$. A subtwistor structure on a Lie group G is called *left-invariant* (*right-invariant*) if it is invariant under left (right) translations by elements of G. A left-invariant (right-invariant) subtwistor structure on a Lie group G is called *isotropic-degenerated* if its radical contains the isotropy subalgebra $\mathfrak{h}$. We proved in [1] that the set of all G-invariant subtwistor structures on a homogeneous space $M = G/H$ is in a one-to-one correspondence with the set of all G-left-invariant H-right-invariant isotropic-degenerated subtwistor structures on the Lie group G. Thus, the existence on a Lie group G of a G-left-invariant H-right-invariant isotropic-degenerated subtwistor structure with radical of rank r implies the existence of a G-invariant subtwistor structure with radical of rank r on M and vice versa. The definition of G-invariant subtwistor structure implies that such subtwistor structure is completely determined by its value at the origin point o. Proposition 4.2 yields

Proposition 6.1. *Let $M = G/H$ be a homogeneous space of dimension $n \geq 3$. The set of all G-invariant subtwistor structures with radical of rank $r = n - 2k$ on M is isomorphic to $\mathrm{gr}^{2k} \times \mathrm{SO}(2k)/\mathrm{U}(k) \times \mathrm{SM}(r)$, where gr^{2k} is the $2k$-Grassmannian in $\mathbb{R}^n$, while $\mathrm{SM}(r)$ is the set of all nondegenerate positive-defined symmetric $r \times r$-matrices.*

This assumption implies that the subtwistor subbundle of G-invariant subtwistor structures over a homogeneous space is trivial even if the subtwistor bundle itself is nontrivial. However, the subbundle of all G-invariant subtwistor bundles over a homogeneous space can be empty.

Theorem 6.2. *Let $M = G/H$ be a homogeneous space of dimension ≥ 3. If the isotropy representation acts irreducibly on T_oM then M does not admit G-invariant subtwistor structures with nontrivial radical.*

Proof. Suppose that M admits a G-invariant subtwistor structure (Ω, D, Φ, β) with nontrivial radical. Since the work bundle D is a G-invariant distribution on M and dh is an automorphism of the tangent space T_oM for every $h \in H$, we infer that dh is an automorphism of D_o for all $h \in H$. Thus, D_o is a nontrivial invariant subspace for the isotropy action, which contradicts the irreducibility of the isotropy action on T_oM.

Remark 6.3. Theorem 2.2 implies that there are no G-invariant skew-symmetric 2-forms with trivial radical on a homogeneous space G/H of odd dimension. Reckoning with Theorem 6.2, we conclude that a homogeneous space of odd dimension with an irreducible action of the isotropy representation admits no subtwistor structures.

It is known that there are no symplectic structures for $n \geq 3$ on the sphere S^n (see [10]). Let us extend this fact to all subtwistor structures with closed fundamental 2-form.

Theorem 6.4. *The sphere S^n, $n \geq 3$, admits no subtwistor structures with closed fundamental 2-form and hence no sub-Kähler structures.*

Proof. The sphere S^n can always be presented as a homogeneous Riemannian space G/H with irreducible isotropy action (see [7]). Suppose that S^n has a subtwistor structure (Ω, D, Φ, β): $d\Omega = 0$. The value of this subtwistor structure at the origin point o induces the left-invariant isotropic-degenerated subtwistor structure $(\Omega_o \circ \tau, \tau^{-1}(D_o), \tau^{-1} \circ \Phi_o \circ \tau, \beta \circ \tau)$ on the Lie group G. Since the isotropy subgroup of a homogeneous Riemannian space is compact, applying the averaging operation over the subgroup H (integration with respect to the unimodular measure) to this subtwistor structure, we obtain the G-left-invariant H-right-invariant isotropic-degenerated subtwistor structure on the Lie group G with radical of rank at least $\operatorname{rad}\Omega$. The so-obtained subtwistor structure on the Lie group G induces a G-invariant subtwistor structure on M with closed fundamental 2-form Ω'. Since S^n has trivial second cohomology group $H^2(S^n, \mathbf{z})$, on S^n there is a nonzero G-invariant 1-form α': $d\alpha' = \Omega'$. On the other hand, we proved in [3] that the sphere S^n, $n \geq 2$, admits no nonzero G-invariant 1-forms. The so-obtained contradiction proves that S^n admits no subtwistor structures with closed fundamental 2-form.

Since S^n is a homogeneous space with an irreducible action of the isotropy subgroup (see [7]), Theorem 6.2 yields

Corollary 6.5. *The sphere S^n, $n \geq 3$, admits no invariant subtwistor structures with nontrivial radical.*

If the isotropy subgroup H of a homogeneous space $M = G/H$ is a normal subgroup in G then G/H is the quotient Lie group. Since each Lie group is a parallelizable manifold, Proposition 4.5 gives a class of parallelizable manifolds admitting a G-invariant subtwistor structure.

Corollary 6.6. *Let $M = G/H$ be a homogeneous space of dimension $n \geq 3$ and let H be a normal subgroup in G. Then the subtwistor bundle over M is trivial and each subtwistor structure in $\mathbb{R}^n$ generates a G-invariant subtwistor structure on M.*

Let X be a vector field on a homogeneous space $M = G/H$. We call a vector field X *equivariant* if $dg^{-1}X(g(o)) = X(o)$ for all $g \in G$. Note that the mapping τ is a homomorphism of the Lie bracket of elements in the subspace $\mathfrak{m}$ and the Lie bracket of equivariant vector fields on M. Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ be the first derived ideal of the Lie algebra $\mathfrak{g}$.

Proposition 6.7. *Suppose that $M = G/H$ is a homogeneous Riemannian space of dimension ≥ 3, the Lie algebra $\mathfrak{g}$ has nontrivial center $\mathfrak{c}$, and $\mathfrak{p} = \mathfrak{m} \cap \mathfrak{c} \cap \mathfrak{g}' \neq \{0\}$. Then each element of $\mathfrak{p}$ generates a G-invariant subtwistor structure with closed fundamental 2-form on M.*

Proof. For a homogeneous Riemannian space $M = G/H$, the group G is the isometry group of the Riemannian metric B on M; i.e., the metric B is a G-invariant bilinear form. Let ξ be an equivariant vector field on M. This ξ generates the nonzero 1-form α on M: $\alpha(X) = B(\xi, X)$, $X \in C^1(TM)$. Put $x = g(o)$, $g \in G$. For every vector field X on M, we have

$$\alpha_x(dgX) = B_x(\xi, dgX) = B_o(dg^{-1}\xi, X) = B_o(\xi, X) = \alpha_o(X);$$

i.e., α is G-invariant. Then $d\alpha$ is a G-invariant skew-symmetric closed 2-form on M. If $\tau^{-1}(\xi) \in \mathfrak{g}' \setminus \{0\}$ then there are $X, Y \in \mathfrak{g}$ such that $\tau^{-1}(\xi) = [X, Y]$. Then

$$2\, d\alpha(\tau X, \tau Y) = -\alpha([\tau X, \tau Y]) = -B(\xi, \tau[X, Y]) = -B(\xi, \xi) \neq 0,$$

i.e., $d\alpha \not\equiv 0$.

Let D be the orthogonal complement to $\operatorname{rad}(d\alpha)$ with respect to the metric B. In [3], we proved that D is a G-invariant distribution on M. Each complex structure J in the fibers of D orthogonal with respect to the metric B such that $d\alpha(X, Y) = B(JX, Y)$ for all $X, Y \in C^1(D)$ generates a G-invariant affinor Φ associated with $d\alpha$. We have obtained a G-invariant subtwistor structure $(d\alpha, D, \Phi, \beta)$, where β is the radical metric obtained by restricting B to $\operatorname{rad}(d\alpha)$. It remains to prove that each nonzero element in $\mathfrak{p}$ generates an equivariant vector field on M.

Take $X \in \mathfrak{p} \setminus \{0\}$ and let G_t be the one-parameter subgroup generated by X. Denote by X^* the vector field on M such that $X^*(x) = \frac{d}{dt}|_{t=0} G_t(x)$ for every $x \in M$. Since G_t lies in the center of G, for every $g \in G$ we have

$$dg^{-1}X^*(g(o)) = \frac{d}{dt}\bigg|_{t=0} g^{-1}G_t g(o) = \frac{d}{dt}\bigg|_{t=0} G_t(o) = X^*(o).$$

Thus, X^* is an equivariant vector field on M generating a G-invariant closed 1-form on M.

Proposition 6.7 enables us to obtain a class of homogeneous spaces with invariant subtwistor structure. Since every nilpotent Lie group has nontrivial center and, given a compact subgroup H, we can construct an H-bi-invariant Riemannian metric (see [7]) for the homogeneous space $M = G/H$, where G is a nilpotent Lie group, while H is a proper compact subgroup in G transversal to the center of G; therefore, we can construct a G-invariant subtwistor structure with exact fundamental 2-form. Moreover, an invariant subtwistor structure with exact fundamental 2-form on a homogeneous space can be obtained directly from an invariant affinor metric structure provided that a 1-form is considered instead of a fundamental 2-form; these structures are studied in [3]. Note that if a homogeneous space $M = G/H$ admits a G-invariant subtwistor structure and M is a compact manifold without boundary then this subtwistor structure has radical of nonzero rank because every exact skew-symmetric 2-form on a compact orientable manifold without boundary is degenerate. Theorem 2.2 also implies that if a homogeneous space of odd dimension admits an invariant subtwistor structure then this structure has radical of dimension ≥ 1. The semidirect product of a symplectic Lie group and a commutative Lie group provides an easy example of a Lie group with left-invariant subtwistor structure having closed fundamental 2-form and a nontrivial radical.

Remark 6.8. If a Lie group G contains a proper subgroup H of even codimension and admits a G-left-invariant H-right-invariant subtwistor structure with closed fundamental 2-form and radical coinciding with the isotropy subalgebra $\mathfrak{h}$ then this subtwistor structure induces a G-invariant almost Kähler structure on the homogeneous space $M = G/H$. In the case that the torsion tensor of this subtwistor structure is equal to zero, this tensor induces a G-invariant Kähler structure on M.

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