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DEGENERATE MULTILINEAR FORMS AND HERMITIAN AND PARA-HERMITIAN STRUCTURES

E. S. Kornev UDC 514.763

Abstract: We describe some method for obtaining families of complex and paracomplex structures on real manifolds by using degenerate skew-symmetric multilinear forms. To construct these structures, we employ a skew-symmetric form with nontrivial radical and obtain a family of almost complex structures on the six-dimensional sphere different from the Cayley structure and families of Hermitian and para-Hermitian structures on some six-dimensional manifolds.

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1. Introduction

The problem of constructing a complex structure, a Hermitian structure, or a Kähler structure is well known for real manifolds of even dimension. This structure makes it possible to define complex coordinates and a Hermitian metric on a real manifold. A Hermitian structure on a real manifold M of dimension $2n$ is a pair (J, h), where h is a Riemannian metric on M and J is a complex structure on M preserving h. There are well known examples both of manifolds admitting or lacking a Hermitian structure. The notion of a para-Hermitian structure was introduced later on real manifolds of dimension $2n$ with a pseudo-Riemannian metric . A para-Hermitian structure on a real manifold M of dimension $2n$ is a pair (Φ, g), where g is a pseudo-Riemannian metric of signature (n, n) on M and Φ is a paracomplex structure on M such that $g \circ \Phi = -g$. Some survey of the available properties and results for para-Hermitian structures can be found in [1, 2]. As regards, Hermitian and para-Hermitian structures, the notion is defined of a fundamental 2-form which is a nondegenerate skew-symmetric bilinear form on the manifold. Hermitian and para-Hermitian structures with closed fundamental form are called *Kähler* (*para-Kähler*) *structures*. In [3, 4], the notions of a complex structure, a Kähler structure, and a contact structure were generalized for degenerate 1- and 2-forms on real manifolds of any dimension or vector bundles. These are *subtwistor structures*, *sub-Kähler structures*, and *affinor metric structures.* In [5], some method is given for constructing an almost complex structure on six-dimensional manifolds by means of a nondegenerate skew-symmetric 3-form but [5] does not address the skew-symmetric 3-forms with nonzero degeneration set which we will call the *radical.*

The main goal of the present article is to describe the method for constructing a Hermitian structure or a para-Hermitian structure on a real manifold M of dimension $2n$ using a degenerate skew-symmetric n-form with radical of rank n on M. This method makes it possible to obtain Hermitian and para-Hermitian structures on the manifolds where it was impossible to construct the structures by other methods. For instance, it was shown in [6] that an almost complex structure on each six-dimensional manifold in the seven-dimensional Euclidean space obtained by multiplying Cayley octaves is not integrable. In [7], there were constructed integrable complex and paracomplex Cayley structures on six-dimensional pseudospheres. However, it was not proved in [6, 7] that the six-dimensional sphere or a pseudosphere as well as the six-dimensional product of spheres does not admit integrable almost complex structures different from the Cayley structures. Using a degenerate skew-symmetric form on an even-dimensional manifold, we obtain a family of almost Hermitian structures on the six-dimensional sphere different from the Cayley structure and a family of Hermitian structures on the direct product of a two-dimensional sphere and

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a four-dimensional sphere. For constructing these families, we use the relationship between Hermitian and para-Hermitian structures on a manifold M and degenerate multilinear forms which is described in Section 4.

Section 2 contains the needed information and results from the theory of Hermitian and para-Hermitian structures. In Section 3, we obtain and describe some results for the radical of a degenerate skew-symmetric multilinear form. In Section 4, we describe the relation between Hermitian (para-Hermitian) structures and skew-symmetric multilinear forms with nontrivial radical. In Section 4, we prove the main theorem, which states that the existence of a complex structure or a paracomplex structure on a manifold of dimension $2n$ is equivalent to the existence of a closed exterior n-form with radical of rank n. In Section 5, we construct a family of almost complex structures on the six-dimensional sphere different from the Cayley structure and give criteria for the integrability of the structures. It is also proved in Section 5 that the six-dimensional sphere admits no almost paracomplex structures. In Section 6, we study the existence of Hermitian and para-Hermitian structures on other six-dimensional manifolds. In particular, we prove that on the direct product of the two-dimensional sphere and the four-dimensional sphere, there is a family of Hermitian structures but there are no para-Hermitian structures.

2. Hermitian and Para-Hermitian Structures

In this section, we give the main notions and information from the theory of Hermitian and para-Hermitian structures. The detailed surveys of results for Hermitian and para-Hermitian structures can be found in [1, 2] and also in [8, Chapter 9].

Let M be a real manifold of class C^∞ of dimension $2n$, let h be a Riemannian metric on M, and let g be a pseudo-Riemannian metric on M of signature (n,n). We will denote the tangent bundle over M by TM and the cotangent bundle over M, by T^*M. Refer as an *almost complex structure* on M to a continuous field J of automorphisms of the tangent spaces on M such that $J^2 = -\operatorname{id}$, where id is the field of identity linear operators in the fibers of the tangent bundle TM. An *almost paracomplex structure* on M is a continuous field Φ of automorphisms of the tangent spaces on M such that $\Phi^2 = \operatorname{id}$ and the rank of the subbundles of eigenspaces for the eigenvalues ± 1 is equal to n. An *almost Hermitian structure* on M is a pair (J,h), where J is an almost complex structure on M such that $h \circ J = h$. An *almost para-Hermitian structure* on M is a pair (Φ, g), where Φ is an almost paracomplex structure on M such that $g \circ \Phi = -g$.

The *complexification* of a real vector space V is the complex vector space $V_{\mathbb{C}} = V \otimes \mathbb{C}$. The complex structure in the real vector space V has no eigenvalues but has two eigenvalues $\pm i$, $i = \sqrt{-1}$ in the complexified vector space $V_{\mathbb{C}}$. An almost complex structure defines a complex structure in the complexification of the tangent space T_xM, and an almost paracomplex structure defines a paracomplex structure in the tangent space T_xM at every $x \in M$. However, not all these structures admit complex or paracomplex local coordinates consistent with the action of these structures on local vector fields. An almost complex structure J is *integrable* or *complex* if for every $x \in M$ there exist local real coordinates $(x_1, \dots, x_n, y_1, \dots, y_n) : \partial y_k = J\partial x_k$ for all $k = 1, 2, \dots, n$, where ∂x_k stand for the local basis vector fields, acting at a smooth function f as $\partial x_k(f) = \frac{\partial f}{\partial x_k}$. Similarly we define the notion of an integrable almost paracomplex structure called a *paracomplex structure*. An almost Hermitian structure (J,h) is *Hermitian*, and an almost para-Hermitian structure (Φ, g) is *para-Hermitian* if the almost complex structure J or the almost paracomplex structure Φ is integrable.

A distribution D of tangent subspaces on a manifold M is *regular* if $\dim(D(x)) = \text{const}$ on M and for every $x \in M$ there exists an open neighborhood U such that in U there exists a continuous local basis for $D|_U$. An irregular distribution is a *singular distribution*.

Let (J,h) be an almost Hermitian structure on a manifold M. Consider the complex vector bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$. Since the almost complex structure J extends to $T_{\mathbb{C}}M$ and has only the eigenvalues $\pm i$, $i = \sqrt{-1}$, we obtain $T_{\mathbb{C}}M = V_+ \oplus V_-$, where V_+ is the distribution of eigenspaces with eigenvalue i, while V_- is the distribution of eigenspaces for eigenvalue $-i$. The characteristic polynomial for J has the form $(z^2+1)^n$, whence $\dim(V_+(x)) = \dim(V_-(x)) = n$ at every $x \in M$. Extend the metric h to $T_{\mathbb{C}}M$ by assuming that

$$h(\lambda X, \mu Y) = \lambda\overline{\mu}\, h(X, Y)$$

for all real vector fields $X, Y \in C^\infty(TM)$ and complex functions $\lambda, \mu \in C^\infty(M)$, where $\overline{\mu}$ stands for complex conjugation. The condition $h \circ J = h$ implies that the distributions of V_+ and V_- are orthogonal in the metric h. It follows that to every almost Hermitian structure there corresponds the pair of orthogonal distributions of V_+ and V_-. Observe that these distributions can be regular or singular. Conversely, if the bundle $T_{\mathbb{C}}M$ is the direct sum of regular distributions of V_+ and V_- of rank n then this pair of distributions defines the almost complex structure

$$J : J|_{V_+} = i\,\mathrm{id}, \quad J|_{V_-} = -i\,\mathrm{id}\,.$$

Since each real vector field $X \in C^\infty(TM)$ is representable as

$$X = Z_+ + Z_-, \quad Z_+ \in C^\infty(V_+),\ Z_- \in C^\infty(V_-),$$

the value of the almost complex structure J at a real vector field X is defined as follows:

$$JX = i(Z_+ - Z_-).$$

If a Riemannian metric h_0 is chosen on M and an almost complex structure J does not preserve the metric H_0 then we can always construct a metric $h = h_0 + h_0 \circ J$ for which $h \circ J = h$. Thus, defining an almost Hermitian structure on a manifold M is equivalent to defining a Riemannian metric and a pair of regular distributions of V_+ and V_- such that

$$T_{\mathbb{C}}M = V_+ \oplus V_-, \quad \mathrm{rank}(V_+) = \mathrm{rank}(V_-).$$

The Complex Frobenius Theorem and the definition of V_+ and V_- imply the following (see [8, Chapter 9]):

Proposition 2.1. *An almost Hermitian structure on a real even-dimensional manifold M is Hermitian if and only if the distributions of the eigenspaces V_+ and V_- are involutive.*

Thus, for defining a Hermitian structure on a manifold M, it suffices to endow M with a Riemannian metric and a pair of involutive regular distributions of the same rank whose direct sum is equal to $T_{\mathbb{C}}M$.

Let (Φ, g) be an almost para-Hermitian structure on a manifold M. As in the case of an almost Hermitian structure, the tangent bundle TM can be split into a direct sum of real distributions D_+ and D_-; i.e., $\Phi|_{D_+} = \mathrm{id}$ and $D_-{:}\Phi|_{D_-} = -\,\mathrm{id}$. The condition $g \circ \Phi = -g$ implies that D_+ and D_- are the distributions of the maximal isotropic subspaces. Conversely, if M is endowed with a pseudo-Riemannian metric g_0 of signature (n, n) and a pair of regular distributions of D_+ and D_- such that

$$TM = D_+ \oplus D_-, \quad \mathrm{rank}(D_+) = \mathrm{rank}(D_-) = n,$$

then D_+ and D_- define an almost paracomplex structure Φ for which they are the distributions of eigenspaces with eigenvalues ± 1. If the metric g_0 does not satisfy the condition $g_0 \circ \Phi = -g_0$ then the latter will be fulfilled for the metric $g = g_0 - g_0 \circ \Phi$, and we obtain a para-Hermitian structure (Φ, g). The Real Frobenius Theorem and the definition of distributions D_+ and D_- imply the following (see [1]):

Proposition 2.2. *An almost para-Hermitian structure on a real even-dimensional manifold M is para-Hermitian if and only if the distributions of the eigenspaces D_+ and D_- are involutive.*

Thus, for defining a para-Hermitian structure on a manifold M, it suffices to endow M with a pseudo-Riemannian metric of signature (n, n) and a pair of involutive distributions of identical rank whose direct sum is equal to TM.

Remark 2.3. Since using a partition of unity, on a paracompact manifold we can construct a Riemannian metric (see [9]) or a pseudo-Riemannian metric, on a paracompact manifold M of dimension $2n$, for defining an almost Hermitian structure or para-Hermitian structure, it suffices to define some decomposition of the complexified or real tangent bundle TM into the direct sum of distributions of rank n. Likewise, for defining a Hermitian or a para-Hermitian structure on a paracompact manifold M, it suffices to define some decomposition of the complexified of real tangent bundle TM into the direct sum of involutive regular distributions of rank n.

3. The Radical of a Multilinear Form

Suppose that M is a real manifold of class C^∞ of dimension $n \geq 3$, while p is a positive integer and Ω is a p-linear nonzero form on M. Denote by $\mathrm{I}_X \Omega$ the $(p-1)$-linear form on M obtained by substituting a vector field X for the first argument in the multilinear form Ω. This $(p-1)$-linear form is the *interior product of X and Ω.*

Definition 3.1. If $p \geq 2$ then the *radical of a p-linear form Ω at $x \in M$* is the vector space

$$\operatorname{rad}\Omega_x = \{v \in T_x M : \mathrm{I}_v \Omega_x = 0\}.$$

For $p = 1$, the *radical* of a 1-*form* Ω is the radical of its exterior differential $d\Omega$ at x. The distribution $\operatorname{rad}\Omega = \bigcup_{x \in M} \operatorname{rad}\Omega_x$ on M is the *radical of the p-linear form Ω on M.*

A p-linear form Ω on a manifold M is *regular* if $\operatorname{rad}\Omega$ is a regular distribution on M.

Observe that the radical of the bilinear form Ω is sometimes called the *kernel* of Ω. The radical and kernel of a bilinear form are the same, but for 1-forms they are differ.

We say that a skew-symmetric p-form Ω, with $p \geq 2$, is *nondegenerate on a manifold M* if $\operatorname{rad}\Omega = \{0\}$. For a nondegenerate skew-symmetric 2-form Ω on a manifold M of dimension $2n$, we obtain the familiar property: $\Omega^n \neq 0$ on M, and for a nondegenerate 1-form η on a manifold M of dimension $2n+1$, we get that $(d\eta)^n \wedge \eta \neq 0$ on M (a contact 1-form). Definition 3.1 implies immediately that on a manifold of dimension 2, any regular nonzero skew-symmetric 2-form is nondegenerate. Note that for a regular skew-symmetric multilinear form Ω with nontrivial radical, the restriction of Ω to any distribution complementary to $\operatorname{rad}\Omega$ is always nondegenerate.

We now obtain some important properties for the rank of the radical of a regular skew-symmetric p-form.

Proposition 3.2. *Let M be a paracompact manifold of dimension $n \geq 3$. Then* $\operatorname{rank}(\operatorname{rad}\Omega) \leq n - p$ *for every nonzero regular skew-symmetric p-form Ω on M.*

Proof. Since a paracompact manifold admits a Riemannian metric (see Remark 2.3), the distribution $\operatorname{rad}\Omega$ on M has the orthogonal complement D. Denote by X^0 the projection of the vector field X to $\operatorname{rad}\Omega$ and designate as X' the projection of X to D. Given $X_1, \dots, X_p \in C^\infty(TM)$, we have

$$\Omega(X_1, \dots, X_p) = \Omega\big(X_1^0 + X_1', \dots, X_p^0 + X_p'\big) = \Omega(X_1', \dots, X_p').$$

Since every collection of m vectors in a vector space of dimension less that m is always linearly dependent, for every $x \in M$ we obtain $\Omega_x(X_1', \dots, X_p') = 0$ for $p > m = n - \operatorname{rank}(\operatorname{rad}\Omega)$. Since Ω is a regular nonzero p-form; therefore, $\operatorname{rank}(\operatorname{rad}\Omega) \leq n - p$. $\square$

Denote the Lie bracket of vector fields X, Y on a manifold M by $[X, Y]$.

Proposition 3.3. *The radical of any closed regular skew-symmetric p-form, $p \geq 2$, on a manifold M is an involutive regular distribution on M.*

Proof. Use the definition of the exterior derivative of a skew-symmetric p-form Ω. Given $X_1, \dots,$ $X_{p+1} \in C^\infty(TM)$, we infer

$$(p+1)!\, d\Omega(X_1, \dots, X_{p+1}) = \frac{1}{(p+1)!}\Bigg(\sum_{k=1}^{p+1} (-1)^{k-1} X_k(\Omega(X_1, \dots, \widehat{X}_k, \dots, X_{p+1})) + \sum_{k<l} (-1)^{k+l} \Omega([X_k, X_l], \dots, \widehat{X}_k, \dots, \widehat{X}_l, \dots, X_{p+1})\Bigg), \qquad (1)$$

where $\widehat{X}_k$ stands for the omission of X_k. Definition 3.1 implies for all $X_1, X_2 \in C^\infty(\operatorname{rad}\Omega)$ and $X_3, \dots, X_{p+1} \in C^\infty(TM)$ that $X_k(\Omega(X_1, \dots, \widehat{X}_k, \dots, X_{p+1})) = 0$ for each index k and

$$\Omega([X_k, X_l], \dots, \widehat{X}_k, \dots, \widehat{X}_l, \dots, X_{p+1}) = 0$$

for $k \neq 1$ and $l \neq 2$. Since $d\Omega = 0$, we have

$$\Omega([X_1, X_2], X_3, \dots, X_{p+1}) = 0,$$

i.e., $[X_1, X_2] \in C^\infty(\operatorname{rad}\Omega)$. $\square$

The following example shows that the converse to Proposition 3.3 fails.

Example 3.4. Consider the compact manifold $M = S^6 \times T^n$, where S^6 is the six-dimensional sphere, and T^n is the flat n-dimensional torus. The sphere S^6 admits an almost Hermitian structure with fundamental 2-form Ω_0 (see [6]). Since there are no symplectic structures on the six-dimensional sphere and the fundamental 2-form of an almost Hermitian structure is always nondegenerate, Ω_0 is a nonclosed 2-form on S^6. Extend the 2-form Ω_0 to some skew-symmetric 2-form Ω on M by setting

$$\Omega(X,Y)|_{X,Y\in T(S^6)} = \Omega_0(X,Y), \quad \Omega(X,Y)|_{X\in T(S^6),Y\in T(T^n)} = 0, \quad \Omega(X,Y)|_{X,Y\in T(T^n)} = 0.$$

We obtain $\operatorname{rad}\Omega = T(T^n)$, $d\Omega \neq 0$, and $\operatorname{rad}\Omega$ is an involutive distribution on M.

It was shown in [4] that, for every regular degenerate skew-symmetric 2-form Ω on a manifold M of an arbitrary dimension, each distribution on M complementary to $\operatorname{rad}\Omega$ has even rank and the restriction of the 2-form Ω to this distribution is nondegenerate. By the Frobenius Theorem, an involutive distribution on the manifold M is integrable and hence M is a foliation with integral submanifolds as the leaves. Proposition 3.3 yields

Corollary 3.5. *Let M be a real manifold of dimension $n \geq 3$ and let Ω be a regular closed skew-symmetric 2-form on M with radical of rank $r \geq 1$. Then M is a foliation with leaves of dimension r and any submanifold in M transversal to the leaves is a symplectic submanifold of dimension $n - r$.*

4. Degenerate Skew-Symmetric Multilinear Forms and Hermitian Structures

We will describe the relationship between Hermitian and para-Hermitian structures and closed regular skew-symmetric forms.

Let M be a paracompact real manifold of dimension $2n$. By Remark 2.3, for defining an almost Hermitian structure on M, it suffices to define a Riemannian metric on M and a decomposition of the complexified tangent bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ into the direct sum of regular distributions of complexified tangent subspaces of rank n; and for defining an almost para-Hermitian structure on M, it suffices to define on M a pseudo-Riemannian metric of signature (n, n) and a decomposition of the tangent bundle into a direct sum of regular distributions of tangent subspaces of rank n. If these distributions on M are involutive then we obtain a Hermitian structure or a para-Hermitian structure on M. We will refer as the *complexification* of a skew-symmetric real p-form Ω on M to the extension of Ω to the complexified tangent bundle $T_{\mathbb{C}}M$. Refer as a *complex skew-symmetric p-form* on a real manifold M to a skew-symmetric p-form acting on the sections of the complexified tangent bundle $T_{\mathbb{C}}M$.

Proposition 4.1. *Let M be a real paracompact manifold of dimension $2n$.*

1. *Each regular skew-symmetric real n-form on M with radical of rank n generates an almost para-Hermitian structure on M, while each regular complex n-form on M generates an almost Hermitian structure on M.*
2. *Each regular closed skew-symmetric real n-form on M with radical of rank n generates a para-Hermitian structure on M, while each regular closed skew-symmetric complex n-form on M generates a Hermitian structure on M.*

Proof. Let Ω be a complex regular skew-symmetric n-form on M with radical of rank n. Since a paracompact manifold always has a Riemannian metric, choose a Riemannian metric h and extend it to $T_{\mathbb{C}}M$. Denoting by D the orthogonal complement to the distribution $\operatorname{rad}\Omega$ with respect to h, we have

$$T_{\mathbb{C}}M = D \oplus \operatorname{rad}\Omega, \quad \operatorname{rank}(D) = \operatorname{rank}(\operatorname{rad}\Omega) = n.$$

Defining the almost complex structure and the Riemannian metric as in Section 2, we obtain an almost Hermitian structure on M. If the n-form Ω is closed, then Proposition 3.3 implies that $\operatorname{rad}\Omega$ is involutive. Without loss of generality, we may assume that $\operatorname{rad}\Omega$ is a distribution of vector fields of type (1,0). Since the involutivity of a distribution of vector fields of type (1,0) is equivalent to the integrability of the almost Hermitian structure (see [8, Chapter 9]), we obtain the second part of items 1 and 2.

Since on a paracompact manifold of dimension $2n$, there always exists a pseudo-Riemannian metric of signature (n, n), we similarly obtain the first part of items 1 and 2. $\square$

We now prove the converse of Proposition 4.1:

Theorem 4.2. *Let M be a paracompact real manifold of dimension $2n$. Each almost Hermitian structure on M defined by a pair of regular distributions of $V_+, V_- \subset T_{\mathbb{C}}M$ of rank n generates a complex regular n-form on M with radical V_+ which is closed if the structure is Hermitian; and each para-Hermitian structure on M defined by a pair of real regular distributions of D_+ and D_- of rank n generates a real regular n-form with radical D_+ on M which is closed if the structure is para-Hermitian.*

Proof. Identify the real space $\mathbb{R}^{2n}$ with the complex space $\mathbb{C}^n$. If M admits an almost Hermitian structure then for M there exists an open covering $\{U\}_{\alpha\in A}$, where U_α is an open set in M diffeomorphic to an open ball in $\mathbb{C}^n$. Let $V_+, V_- \subset T_{\mathbb{C}}M$ be the distributions of eigenspaces with eigenvalues i and $-i$ respectively. On every open set U_α there exists a collection of linearly independent complex 1-form $\xi_1^\alpha, \dots, \xi_n^\alpha$ so that

$$V_+|_{U_\alpha} = \bigcap_{k=1}^{n} \ker \xi_k^\alpha.$$

Since the kernel of a nonzero linear functional has codimension 1, in $C^\infty(V_-|_{U_\alpha})$ there exist sections $Z_1^\alpha, \dots, Z_n^\alpha$ so that $\xi_k^\alpha(Z_k^\alpha) = 1$ and $\xi_l^\alpha(Z_k^\alpha) = 0$ for $k \neq l$. It follows that

$$\xi_1^\alpha \wedge \cdots \wedge \xi_n^\alpha \big(Z_1^\alpha, \dots, Z_n^\alpha\big) = \frac{1}{n!}.$$

The Partition-of-Unity Theorem implies that for each index $\alpha \in A$ there exists a function $\phi_\alpha \in C^\infty(M)$ so that $0 < \phi_\alpha(x) \leq 1$ for all $x \in U_\alpha$, $\phi_\alpha(x) = 1$ on some closed subset $\overline{V}_\alpha \subset U_\alpha$, and $\phi_\alpha(x) = 0$ for all $x \in M \setminus U_\alpha$. Then $\Omega_\alpha = \phi_\alpha \xi_1^\alpha \wedge \cdots \wedge \xi_n^\alpha$ be a skew-symmetric n-form on U_α with radical $V_+\big|_{U_\alpha}$. Put $\Omega = \sum_{\alpha\in A} \Omega_\alpha$. Since M is paracompact, every $x \in M$ belongs to finitely many intersections of U_α, and hence the sum is finite at every point and Ω is a skew-symmetric n-form on M. Observe that $\operatorname{rad}(\omega + \theta) = V_+$ for all p-forms ω and θ so that $\operatorname{rad}\omega = \operatorname{rad}\theta = V_+$. Since $\operatorname{rad}\Omega_\alpha = \operatorname{rad}\Omega_\beta = V_+|_{U_\alpha \cap U_\beta}$, for all α and β so that $U_\alpha \cap U_\beta \neq \varnothing$, we have $\operatorname{rad}\Omega = V_+$.

For a Hermitian structure, V_+ is an involutive distribution of holomorphic vector fields and V_- is an involutive antiholomorphic distributions of vector fields. Using these facts and equality (1) of Section 3, we obtain $\operatorname{rad}\Omega \subset \operatorname{rad}(d\Omega)$. If $d\Omega \neq 0$ then $\operatorname{rank}(\operatorname{rad}(d\Omega)) \geq n$. On the other hand, Proposition 3.2 implies that $\operatorname{rank}(\operatorname{rad}(d\Omega)) \leq n - 1$. Consequently, $d\Omega = 0$. The property $d\Omega = 0$ can also be proved using the fact that for every $x \in M$ there exists a closed neighborhood $\overline{V}_\alpha$ so that $\Omega|_{\overline{V}_\alpha} = \xi_1^\alpha \wedge \cdots \wedge \xi_n^\alpha$. This is because the local 1-forms $\xi_1^\alpha, \dots, \xi_n^\alpha$ can always be chosen exact.

Likewise, for an almost para-Hermitian structure on M, we can construct a real regular n-form with radical D_+ which will be closed for a para-Hermitian structure. $\square$

Proposition 4.1 and Theorem 4.2 yield

Corollary 4.3. *Let M be a paracompact real manifold of dimension $2n$. Then M admits a Hermitian structure with regular distributions of eigenspaces if and only if M admits a complex regular skew-symmetric closed n-form with radical of rank n; M admits a para-Hermitian structure with regular distributions of eigenspaces if and only if M admits a real regular skew-symmetric closed n-form with radical of rank n.*

The following result makes it possible to construct an example of a Hermitian structure or a para-Hermitian structure obtained by a regular skew-symmetric n-form with radical of rank n.

Proposition 4.4. *Let M be a paracompact real manifold of dimension $2n$. If M has a global closed n-coframe $\xi_1, \dots, \xi_n$ such that $d\xi_k = 0$ for all $k \le n$ then M admits a Hermitian structure and a para-Hermitian structure.*

PROOF. Consider the global skew-symmetric n-form $\Omega = \xi_1 \wedge \dots \wedge \xi_n$ on M. It is easy to see that $d\Omega = 0$ and $\operatorname{rad}\Omega = \bigcap_{k=1}^n \ker \xi_k$. From Proposition 4.1 we obtain that the real n-form Ω defines a para-Hermitian structure on M.

Remark 2.3 implies that on a paracompact manifold M, there exists a Riemannian metric. Extend the 1-forms $\xi_1, \dots, \xi_n$ to $T_{\mathbb{C}}M$ by assuming that $\xi_k(\lambda X) = \lambda \xi_k(X)$, $k = 1, 2, \dots, n$, for every $X \in C^\infty(TM)$ and every complex number λ. It is easy to see that $\xi_1, \dots, \xi_n$ are complexified closed 1-forms on M. Then the extension of the n-form Ω to $T_{\mathbb{C}}M$ is a complexified closed n-form with radical of rank n, which by Proposition 4.1 defines a Hermitian structure on M. □

On every real Lie group of dimension $2n$ with the first Betti number $\ge n$, there exists a left-invariant closed n-coframe, and so we obtain

Corollary 4.5. *Suppose that G is a real Lie group of dimension $2n$, $b_1(G)$ is the first Betti number of G, and $b_1(G) \ge n$. Then G admits a left-invariant Hermitian structure and a left-invariant para-Hermitian structure.*

EXAMPLE 4.6. Suppose that G is a real Lie group of dimension $2n$, let $\mathfrak{g}$ be the Lie algebra of G, while $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the first derived ideal in $\mathfrak{g}$ and $\dim(\mathfrak{g}') = n$. In the Lie algebra $\mathfrak{g}$, choose a complement $\mathfrak{p}$ to $\mathfrak{g}'$. Using equality (1) in Section 3 for left-invariant 1-forms, we conclude that $d\alpha = 0$ for every left-invariant 1-form $\alpha \in \mathfrak{p}^*$. Then there exists a left-invariant closed coframe $\xi_1, \dots, \xi_n$ in $\mathfrak{p}^*$. From Proposition 4.4 we conclude that the left-invariant skew-symmetric n-form $\Omega = \xi_1 \wedge \dots \wedge \xi_n$ defines some left-invariant para-Hermitian structure on M and its extension to $\mathfrak{g} \otimes \mathbb{C}$ defines a left-invariant Hermitian structure on G.

Let Ad_g be the adjoint of an element $g \in G$ in $\mathfrak{g}$. It follows from Proposition 3.3 that the distribution $\operatorname{rad}\Omega$ is involutive. Since Ad_g is an isomorphism of $\mathfrak{g}$ with respect to the Lie bracket; $\operatorname{Ad}_g(\operatorname{rad}\Omega)$ for $g \in G$ is a left-invariant involutive distribution on G of rank n. Since the n-form Ω generates a left-invariant para-Hermitian structure on G and its complexification defines a Hermitian structure on G for the distribution $\operatorname{rad}\Omega$, there exists an involutive complement D in $\mathfrak{g}$ or in $\mathfrak{g} \otimes \mathbb{C}$. Then $\operatorname{Ad}_g(D)$ is a left-invariant involutive distribution on G of rank n complementary to $\operatorname{Ad}_g(\operatorname{rad}\Omega)$, and we obtain the left-invariant para-Hermitian or Hermitian structure generated by $g \in G$. Let H be a subgroup in G so that $\operatorname{Ad}_h(\operatorname{rad}\Omega) = \operatorname{rad}\Omega$ for every $h \in H$. Thus, the homogeneous space G/H parametrizes the family of left-invariant para-Hermitian or Hermitian structures obtained as the orbit of the adjoint action of G on the radical of the n-form Ω or as the complexification of the adjoint action of G on the radical of the complexification of the n-form Ω.

Generalizing Example 4.6 for an arbitrary Lie group, we arrive at

Proposition 4.7. *Let G be a real Lie group of dimension $2n$. Each closed left-invariant skew-symmetric n-form Ω on G with radical of rank n generates a family of left-invariant para-Hermitian structures on G parametrized by the points of the orbit of the adjoint action of G on the radical of the n-form Ω, and the complexification of the n-form Ω generates a family of left-invariant Hermitian structures on G parametrized by the points of the orbit of the complexified adjoint action of G on the radical of the complexification of the n-form Ω.*

Let $\Lambda^n(M)$ be the bundle of skew-symmetric n-forms over a manifold M and let $\Lambda^n_{\mathbb{C}}(M)$ be the bundle of complex skew-symmetric n-forms over M. Theorem 4.2 implies that every para-Hermitian structure on a manifold M generates an everywhere no vanishing global section of the bundle $\Lambda^n(M)$, and every Hermitian structure on M is an everywhere no vanishing global section of $\Lambda^n_{\mathbb{C}}(M)$. Let $e(E)$ be the Euler class of a vector bundle E. If E admits an everywhere no vanishing global section then $e(E) = 0$ (see [10]). Thus, we obtain the following necessary condition for the existence of a Hermitian structure or a para-Hermitian structure on a manifold:

Proposition 4.8. *Let M be a real paracompact manifold of dimension $2n$. If M admits an almost para-Hermitian structure then $e(\Lambda^n(M)) = 0$; if M admits an almost Hermitian structure then $e\big(\Lambda^n_{\mathbb{C}}(M)\big) = 0$.*

Let M be a compact boundaryless manifold and let $\chi(M)$ be the Euler characteristic of M. Since $\chi(M) = \int_M e(M)$, where $e(M)$ is the Euler class of the tangent bundle TM (see [10]), we obtain

Corollary 4.9. *Suppose that M is a real paracompact manifold of dimension $2n$ and the bundles $\Lambda^n(M)$ and $\Lambda^n_{\mathbb{C}}(M)$ are compact boundaryless manifolds. If $\chi(\Lambda^n(M)) > 0$ then M admits no para-Hermitian structures; if $\chi\big(\Lambda^n_{\mathbb{C}}(M)\big) > 0$ then M admits no Hermitian structures.*

Proposition 4.10. *If a real paracompact manifold M admits an almost Hermitian structure with distributions of eigenspaces D_+ and D_- then M and the vector bundles D_+ and D_- are orientable.*

PROOF. Since M admits an almost para-Hermitian structure, M must be of dimension $2n$. The pair of distributions of tangent subspaces D_+ and D_- of rank n defines a para-Hermitian structure on M up to multiplication by -1; Theorem 4.2 implies that on M there are real skew-symmetric n-forms Ω_+ and Ω_- such that $\operatorname{rad}\Omega_+ = D_-$ and $\operatorname{rad}\Omega_- = D_+$. Since the restriction of the n-form Ω_+ to the sections of D_+ and the restriction of the n-form Ω_- to the sections of D_- are nondegenerate, these n-forms define a continuous choice of orientation in the fibers of D_+ and D_- respectively, and the skew-symmetric real $2n$-form $\mu = \Omega_+ \wedge \Omega_-$ is a volume form on M. Since on a paracompact manifold, there always exists a Riemannian metric, the bundle D_+ admits a Levi-Civita connection $Q : T(D_+) = Q \oplus D_+$. The lifting of the $2n$-form μ to the sections of Q and the continuous choice of orientation for Ω_+ in the fibers of D_+ generate a volume form on the space of the bundle D_+, i.e., D_+ is an orientable vector bundle. Likewise, we conclude that also D_- is an orientable vector bundle. $\square$

Let is demonstrate how, using degenerate skew-symmetric n-forms, we can obtain some families of Hermitian and para-Hermitian structures. Let Ω be a real or complex regular skew-symmetric n-form with radical of rank n on a real manifold M of dimension $2n$. Denote by A the group of all smooth automorphisms of M. The action of this group at the n-form Ω is defined in the standard manner:

$$a(\Omega) = a^*\Omega = \Omega \circ da, \quad a \in \mathrm{A},$$

where da is the differential of a. Denote by A_Ω the subset

$$\mathrm{A}_\Omega = \{a \in \mathrm{A} : da(\operatorname{rad}\Omega_x) = \operatorname{rad}\Omega_{a(x)}\ \forall x \in M\}.$$

It is easy to check that A_Ω is a subgroup in A. If $d\Omega = 0$ then equality (1) in Section 3 and the properties of the differential of a mapping imply that $d(a^*\Omega) = 0$ for all $a \in \mathrm{A}$. Proposition 4.1 implies that $a^*\Omega$ generates a Hermitian structure or a para-Hermitian structure on M for every $a \in \mathrm{A}\,/\,\mathrm{A}_\Omega$. Thus, we come to

Proposition 4.11. *Let M be a real paracompact manifold of dimension $2n$. If there exists a complex (real) closed skew-symmetric regular n-form with radical of rank n on M then M admits a family of Hermitian (para-Hermitian) structures parametrized by the elements of the orbit of the action of the group of smooth automorphisms of M on the distribution $\operatorname{rad}\Omega$.*

Let M be a real paracompact manifold of dimension $2n$. In [11], the relationship is revealed between complex structures on M and Dirac structures in the bundle $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$. In [12], the relationship is described between paracomplex structures on M and Dirac structures on $TM \oplus T^*M$. Each closed skew-symmetric 2-form B on M generates a transformation of the B-field on M which maps the Dirac structure to another Dirac structure and hence generates a complex structure or a paracomplex structure on M (see [11, 12]). Proposition 4.1 and Theorem 4.2 imply

Proposition 4.12. *Let M be a real paracompact manifold of dimension $2n$. If M admits a complex (real) regular closed skew-symmetric n-form with radical of rank n then M admits a family of Hermitian (para-Hermitian) structures parametrized by the elements of the set of all complex (real) closed skew-symmetric 2-forms on M.*

REMARK 4.13. The families of Hermitian (para-Hermitian) structures of Propositions 4.11 and 4.12 may fail to coincide but always have nonempty intersection.

5. Hermitian and Para-Hermitian Structures on the Six-Dimensional Sphere

Consider the six-dimensional sphere S^6 in the seven-dimensional Euclidean space $\mathbb{R}^7$ which can be identified with the space of imaginary octonions. Given $q_1, q_2 \in \mathbb{R}^7$, we have the skew-symmetric bilinear operation

$$b(q_1, q_2) = \operatorname{Im}(q_1 \overline{q}_2),$$

where $\operatorname{Im}(q)$ is the imaginary part of an octonion q, while $\overline{q}$ stands for the conjugation of q. Since in $\mathbb{R}^7$, there exists a standard Euclidean metric, and the restriction of this metric to S^6 is a Riemannian metric on S^6, each almost complex structure on S^6 generates an almost Hermitian structure on S^6 (see Section 2). Likewise, since the six-dimensional sphere is a compact manifold, by Remark 2.3, each almost paracomplex structure on S^6 generates an almost para-Hermitian structure on S^6. Let G_2 be the group of orthogonal symmetries of the skew-symmetric form b, and let $\mathrm{SU}(3)$ be the group of complex Hermitian matrices with determinant 1. The sphere S^6 can be regarded as the homogeneous space $G_2/\mathrm{SU}(3)$. An almost complex (paracomplex) structure J on S^6 is *G_2-invariant* if $J \circ dg = dg \circ J$ for every $g \in G_2$, where dg is the differential of g. It is known that S^6 admits a nonintegrable G_2-invariant almost complex structure J_0 (see [6]). The distributions of V_+ and V_- of Section 2 for J_0 are G_2-invariant distributions of complex rank 3 and

$$V_+ = \{X - iJ_0X : X \in C^\infty(T(S^6))\}, \quad V_- = \{Y + iJ_0Y : Y \in C^\infty(T(S^6))\}.$$

Let Ω_0 be the fundamental 2-form of the almost Hermitian structure (J_0, h_0), where h_0 is an almost Hermitian metric on S^6. Since the six-dimensional sphere does not admit symplectic structures and Ω_0 is a nondegenerate 2-form of type $(1,1)$, it follows that $d\Omega_0$ is a regular skew-symmetric 3-form on S^6 with zero radical. On the six-dimensional sphere, every real vector field vanishes at least at one point since the Euler characteristic of the six-dimensional sphere is equal to 2.

Proposition 5.1. *On the even-dimensional sphere S^{2n}, there exists a section of the bundle $T_{\mathbb{C}}(S^{2n})$ different from zero at all points of the sphere.*

PROOF. Let p and q be two poles of the sphere S^{2n}, while $U = S^{2n} \setminus p$ and $V = S^{2n} \setminus q$. The stereographic projection centered at q is a diffeomorphism $\phi : U \to \mathbb{R}^{2n}$, and the stereographic projection centered at p is a diffeomorphism $\psi : V \to \mathbb{R}^{2n}$. On $\mathbb{R}^{2n}$, consider the vector field $S(x) = (s_1(x), \dots, S_{2n}(x))$, where $s_k(x) = \exp(-k|x|^2)$, with $|x| = \sqrt{x_1^2 + \dots + x_{2n}^2}$ and $k = 1, 2, \dots, 2n$. The vector field S makes it possible to define some vector fields $X, Y \in C^\infty(T(S^{2n}))$ on S^{2n} such that $X(x) = (d\phi)^{-1}S(\phi(x))$ for $x \in U$, while $X(p) = 0$ and $Y(x) = (d\psi)^{-1}S(\psi(x))$ for $x \in V$ and $Y(q) = 0$. Observe that the vector field X is continuous at p, the vector field Y is continuous at q, and $X|_U \neq 0$, $Y|_V \neq 0$. At each $x \in S^{2n}$, the complex section $Z = X + iY$, with $i = \sqrt{-1}$, of the bundle $T_{\mathbb{C}}(S^{2n})$ is an everywhere no vanishing global section on S^{2n}. $\square$

Extend the Hermitian metric h_0 to a Hermitian inner product on the sections of the bundle $T_{\mathbb{C}}(S^6)$. Since this inner product defines an isomorphism between the sections of $T_{\mathbb{C}}(S^6)$ and complex 1-forms, there exists an everywhere no vanishing complex 1-form $\eta = \mathrm{I}_Z\, h_0$ on S^6, where Z is the complex section of Proposition 5.1. The proof of Proposition 5.1 implies that the complex section $Z = X + iY$ cannot be a section of type (1,0) or of type (0,1) because $Y \neq J_0X$. We conclude that Z and J_0Z are complex sections of $T_{\mathbb{C}}(S^6)$ on S^6 linear independent at every point. We put

$$\Omega_1 = \eta, \quad \omega_2 = \mathrm{I}_{J_0Z}\, h_0, \quad \omega_3 = \mathrm{I}_Z(\mathrm{I}_{J_0Z}\, d\Omega_0).$$

The construction of the 1-form ω_3 implies that $Z, J_0Z \in \ker\omega_3$, and hence the complex 1-forms ω_1, ω_2, and ω_3 are linearly independent at every point in S^6 and $\Omega = \omega_1 \wedge \omega_2 \wedge \omega_3$ is a regular skew-symmetric complex 3-form on S^6 so that

$$\operatorname{rad}\Omega = \ker\omega_1 \cap \ker\omega_2 \cap \ker\omega_3$$

has rank 3. We obtain

Proposition 5.2. *On the six-dimensional sphere S^6, there is a family of complex skew-symmetric regular 3-forms with radical of rank 3 parametrized by the elements of everywhere no vanishing complex 1-forms on S^6.*

Propositions 4.1 and 5.2 imply

Corollary 5.3. *The six-dimensional sphere S^6 admits a family of almost Hermitian structures parametrized by everywhere no vanishing complex 1-forms on S^6.*

Using the above method for constructing a skew-symmetric regular complex 3-form with radical of rank 3 from a complex 1-form on S^6 and Corollary 5.3, we can construct the functional kor : $\omega \mapsto \|d\theta_\omega\|$, where $\|d\theta_\omega\|$ is the norm of the differential of the skew-symmetric 3-form θ_ω constructed from an everywhere no vanishing 1-form ω on S^6. Proposition 4.1 and Theorem 4.2 imply that the almost Hermitian structure corresponding to a 3-form θ_ω is integrable if and only if ω is a zero of the functional kor. We come to

Theorem 5.4. *Suppose that ω is an everywhere no vanishing complex 1-form on the six-dimensional sphere S^6. Suppose further that θ_ω is the skew-symmetric regular complex 3-form on S^6 constructed from this form, and (J_ω, h) is the almost Hermitian structure on S^6 corresponding to the 1-form ω. The following are equivalent:*

(1) *the almost complex structure J_ω is integrable;*

(2) *the 3-form θ_ω is closed;*

(3) *ω is a zero of the functional* kor.

As we see from Theorem 5.4, for obtaining a Hermitian structure on S^6, it suffices to construct a closed complex regular skew-symmetric 3-form with radical of rank 3 on S^6. As was shown above, each everywhere no vanishing complex 1-form ω on S^6 generates a complex 3-form with radical of rank 3:

$$\theta_\omega = \omega_1 \wedge \omega_2 \wedge \omega_3, \quad \omega_1 = \omega.$$

The condition $d\theta_\omega = 0$ gives that this 3-form generates a Hermitian structure on S^6. By now there are no available particular examples of such a closed 3-form on S^6. Here we can only give sufficient conditions for this form to exist on S^6.

Proposition 5.5. *Suppose the fulfillment of one of the following conditions on S^6:*

(1) *there exists a global complex closed 3-coframe on S^6;*

(2) *there exist smooth complex functions f_1, f_2, and f_3 on S^6 whose differentials are linearly independent at every point of S^6;*

(3) *there exists a global complex 3-coframe $\omega_1, \omega_2, \omega_3$ on S^6 so that $d\omega_k \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 = 0$, with $k = 1, 2, 3$.*

Then S^6 admits a Hermitian structure.

Remark 5.6. Proposition 4.11 implies that if there exists a complex regular skew-symmetric closed 3-form with radical of rank 3 on S^6 then it generates a family of Hermitian structures on S^6 parametrized by the elements of the orbit of the action of the group of smooth automorphisms of S^6 on the radical of this 3-form.

For real skew-symmetric 3-forms on S^6, we obtain

Theorem 5.7. *The six-dimensional sphere admits no real skew-symmetric regular 3-forms with radical of rank 3.*

Proof. Suppose that S^6 admits a skew-symmetric regular real 3-form Ω with radical of rank 3. Let $D_+ = \operatorname{rad}\Omega$ and let D_- be the orthogonal complement in the Riemannian metric on S^6 to the distribution of tangent subspaces D_+. Since S^6 is a compact orientable manifold, Proposition 4.10 implies that the distributions of tangent subspaces D_+ and D_- are orientable. Let $e(E)$ be the Euler class of a vector bundle E. Since the Euler class of every vector bundle of odd rank is zero, we have

$$e(S^6) = e(D_+) \smile e(D_-) = 0.$$

But $e(S^6) \neq 0$ since the Euler characteristic of the six-dimensional sphere is equal to 2. Thus, S^6 admits no real regular skew-symmetric 3-forms with radical of rank 3. $\square$

From Theorems 4.2 and 5.7 we obtain

Corollary 5.8. *The six-dimensional sphere admits no almost paracomplex structures.*

6. Hermitian and Para-Hermitian Structures on Six-Dimensional Manifolds

Let us address the existence of Hermitian and para-Hermitian structures on some six-dimensional manifolds and also obtain families of Hermitian and para-Hermitian structures on these manifolds. Note that in [6], almost complex structures on six-dimensional products of spheres were described with the use of octonion multiplication and it was proved that all these almost complex structures are nonintegrable. Here we do not use octonion multiplication and obtain Hermitian structures on using the method of Section 4.

Let S^n be the real sphere in the Euclidean space $\mathbb{R}^{n+1}$ which is a compact simply-connected manifold for every $n \geq 1$. The compact manifold $S^3 \times S^3$ can be embedded in the complex space $\mathbb{C}^4$. Therefore, on $S^3 \times S^3$, there exists a Hermitian structure induced from $\mathbb{C}^4$. Also, since $T(S^3 \times S^3) = T(S^3) \oplus T(S^3)$, this decomposition defines a para-Hermitian structure on $S^3 \times S^3$ (see Section 2). Propositions 4.11 and 4.12 imply

Corollary 6.1. *The direct product of three-dimensional spheres $S^3 \times S^3$ admits a family of Hermitian* (*para-Hermitian*) *structures parametrized by the elements of the orbit of the action of the group of smooth automorphisms of $S^3 \times S^3$ on the radical of the complex* (*real*) *regular closed skew-symmetric* 3*-form on $S^3 \times S^3$ with radical of rank* 3*. Moreover, each complex* (*real*) *closed skew-symmetric* 2*-form on $S^3 \times S^3$ generates a Hermitian* (*para-Hermitian*) *structure on $S^3 \times S^3$.*

Prove that $S^2 \times S^4$ admits a family of Hermitian structures but does not admit a family of para-Hermitian structures.

Theorem 6.2. *On the manifold $S^2 \times S^4$, there exists a family of Hermitian structures parametrized by the set of all complex closed skew-symmetric* 2*-forms on $S^2 \times S^4$.*

Proof. Proposition 5.1 implies that on S^4 there exists an everywhere no vanishing complex section Z of the bundle $T_{\mathbb{C}}(S^4)$. A complex section Z generates a complex distribution of rank 1 on S^4 since $Z \neq 0$ on S^4. The distribution $T_{\mathbb{C}}(S^2) \oplus \mathbb{C}Z$ is an involutive distribution of rank 3 in $T_{\mathbb{C}}(S^2 \times S^4)$. This distribution and its orthogonal complement in a fixed Riemannian metric make it possible to construct a complex structure on $S^2 \times S^4$ as in Section 2. From Theorem 4.2 and Proposition 4.12 we conclude that $S^2 \times S^4$ admits a family of Hermitian structures parametrized by the elements of the space $\bigwedge^2 T^*_{\mathbb{C}}(S^2 \times S^4)$. $\square$

Theorem 6.3. *The direct product of the two-dimensional sphere S^2 and the four-dimensional sphere S^4 does not admit almost para-Hermitian structures.*

Proof. Suppose that on $S^2 \times S^4$ there exists an almost para-Hermitian structure. By Theorem 4.2, this almost para-Hermitian structure generates a real regular skew-symmetric 3-form on $S^2 \times S^4$ with radical of rank 3. As in the proof of Theorem 5.7, we infer that the Euler characteristic of the direct product $S^2 \times S^4$ is equal to zero. But the Euler characteristic of this direct product of spheres is equal to four. Consequently, there are no almost para-Hermitian structures on $S^2 \times S^4$. $\square$

Since on every compact manifold with positive Euler characteristic, any vector field vanishes at some point (see [10]), generalizing Theorem 6.3, we obtain

Theorem 6.4. *Let M be a real compact orientable six-dimensional boundaryless manifold with positive Euler characteristic. Then M admits no para-Hermitian structures.*

Since the product of spheres $S^5 \times S^1$ can be embedded in the complex space $\mathbb{C}^4$, this manifold admits a Hermitian structure. By Theorem 4.2, this Hermitian structure generates a regular complex closed skew-symmetric 3-form on $S^5 \times S^1$ with radical of rank 3. Propositions 4.11 and 4.12 yield

Corollary 6.5. *The direct product of the fifth-dimensional sphere S^5 and the circle S^1 admits a family of Hermitian structures parametrized by the elements of the orbit of the action of the group of smooth automorphisms of $S^5 \times S^1$ on the radical of the complex closed regular skew-symmetric 3-form on $S^5 \times S^1$ with radical of rank 3. Moreover, each complex closed skew-symmetric 2-form on $S^5 \times S^1$ generates a Hermitian structure on $S^5 \times S^1$.*

Theorem 6.6. *The direct product of the fifth-dimensional sphere S^5 and the circle S^1 admits a family of para-Hermitian structures parametrized by the elements of the orbit of the action of the group of smooth automorphisms of $S^5 \times S^1$ on the radical of a real closed regular skew-symmetric 3-form on $S^5 \times S^1$ with radical of rank 3. Moreover, each real closed skew-symmetric 2-form on $S^5 \times S^1$ generates a para-Hermitian structure on $S^5 \times S^1$.*

Proof. We will regard the sphere S^5 as a surface in the complex space $\mathbb{C}^3$:

$$S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}.$$

Let $T^3 = S^1 \times S^1 \times S^1$ be the three-dimensional torus embedded in $\mathbb{C}^3$. Define the action of the torus T^3 at $z \in S^5$ as follows:

$$a(z) = (a_1, z_1, a_2 z_2, a_3, z_3), \quad z = (z_1, z_2, z_3) \in S^5,\ a = (a_1, a_2, a_3) \in S^1 \times S^1 \times S^1.$$

Thus, through every $z \in S^5$, there passes a real three-dimensional submanifold $T^3(z)$. On S^5, introduce the distribution of tangent subspaces D; i.e., $D(z) = T_z(T^3(z))$ for all $z \in S^5$. Since S^5 is a compact manifold, for every $z \in S^5$ there exists an open neighborhood $U : D|_U = T(T^3(z))|_U$. By the Frobenius Theorem, we see that D is a regular involutive distribution of rank 3. Theorem 4.2 implies that on $S^5 \times S^1$, there exists a real closed regular skew-symmetric 3-form with radical D. Now the theorem follows from Propositions 4.11 and 4.12. □

Let M be a real manifold of dimension 4 and let $\Lambda^2(M)$ be the bundle of real skew-symmetric 2-forms over M. $\Lambda^2(M)$ is a real paracompact manifold of dimension 6. Let $\star$ be the Hodge operator on the set of skew-symmetric multilinear forms on M. The properties of the Hodge operator imply that if $\Omega \in \Lambda^2(M)$ then $\star\Omega \in \Lambda^2(M)$, $\star^2 = \mathrm{id}$, and the operator $\star$ has exactly two eigenvalues ± 1. Then the tangent bundle $T(\Lambda^2(M))$ is the Whitney sum of two distributions of tangent subspaces of rank 3 $T(\Lambda^2_+(M))$ and $T(\Lambda^2_-(M))$, where

$$\Lambda^2_+(M) = \{\Omega \in \Lambda^2(M) : \star\Omega = \Omega\}, \quad \Lambda^2_-(M) = \{\Omega \in \Lambda^2(M) : \star\Omega = -\Omega\}.$$

The distributions $T(\Lambda^2_+(M))$ and $T(\Lambda^2_-(M))$ are involutive since they are the tangent bundles for the integral submanifolds $\Lambda^2_+(M)$ and $\Lambda^2(M)$. As was shown in Section 2, this pair of distributions defines a para-Hermitian structure on $\Lambda^2(M)$, and the pair of complexified distributions $T(\Lambda^2_+(M)) \otimes \mathbb{C}, T(\Lambda^2_-(M)) \otimes \mathbb{C}$ defines a Hermitian structure on $\Lambda^2(M)$. Applying Theorem 4.2 and Propositions 4.11 and 4.12, we obtain

Proposition 6.7. *The bundle $\Lambda^2(M)$ of real skew-symmetric 2-forms over a four-dimensional real manifold M admits a family of Hermitian (para-Hermitian) structures parametrized by the elements of the orbit of the action of the group of smooth automorphisms of the manifold $\Lambda^2(M)$ on the radical of a complex (real) closed regular skew-symmetric 3-form on $\lambda^2(M)$ with radical of rank 3. Moreover, each complex (real) closed skew-symmetric 2-form on $\Lambda^2(M)$ generates a Hermitian (para-Hermitian) structure on $\Lambda^2(M)$.*

Remark 6.8. Let M be a real manifold of dimension 4 and let $\Lambda^2_{\mathbb{C}}(M)$ be the bundle of complex skew-symmetric 2-forms over M. Propositions 4.8 and 6.7 imply that the vector bundles $\Lambda^2(M)$ and $\Lambda^2_{\mathbb{C}}(M)$ admit an everywhere no vanishing global section, and have zero Euler class.

Observe that instead of the bundle of all skew-symmetric 2-forms over a four-dimensional manifold M, we can consider the subbundle of only fundamental 2-forms for Hermitian and para-Hermitian structures in the fibers of the tangent bundle TM with fixed metric, in particular, for the twistor bundle.

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E. S. Kornev
Kemerovo State University, Kemerovo, Russia
E-mail address: q148@mail.ru