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Legendrian Submanifolds

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Abstract—The paper studies submanifolds that are tangent at every point to the working distribution of an affinor metric structure. This distribution is a generalization of the contact distribution of a contact metric structure, while an affinor metric structure is a generalization of a contact metric structure to manifolds of arbitrary dimension. Such submanifolds are called Legendrian submanifolds. We consider Legendrian submanifolds of general type and their particular cases: Legendrian curves and homogeneous Legendrian submanifolds.

Keywords: affinor metric structure, 1-form with nontrivial radical, Legendrian submanifold, Legendrian curve, homogeneous space

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INTRODUCTION

In the classical Riemannian geometry on manifolds of odd dimension, mathematicians thoroughly investigated the so called contact metric structures [1]. The affinor metric structure is a generalization of the contact metric structure to manifolds and vector distributions of arbitrary dimension. The affinor metric structures were introduced and described for Lie groups in [2], and they were introduced for Lie algebroids and manifolds in [3]. In the general case, the affinor metric structure is the following set of objects: (α, D, Φ, g), where α is a nonclosed 1-form whose outer differential is a regular outer 2-form, D is a distribution of tangent subspaces such that the restriction of $d\alpha$ to the sections of this distribution is a nondegenerate 2-form, Φ is a continuous field of endomorphims of tangent subspaces associating the Riemannian metric g and the outer differential $d\alpha$. The field of endomorphisms Φ is called the affinor associated with the 1-form α. Affinor metric structures can be considered on both odd-dimensional and even-dimensional manifolds. A distribution D is called the working distribution of an affinor metric structure and can be both holonomic and nonholonomic. In the case when the working distribution D is nonholonomic, the manifold M on which the affinor metric structure is considered contains no integral maximal submanifolds for the distribution D. However, a manifold can contain integral submanifolds for the distribution D of nonmaximal dimension. Such submanifolds are called Legendrian submanifolds. In the contact geometry, mathematicians widely study Legendrian submanifolds and their special case, Legendrian curves. However, the contact metric structure is just a special case of affinor metric structure when the dimension of the manifold is odd and the rank of the radical of the outer differential $d\alpha$ is unity. In this work, Legendrian submanifolds are considered for the most general case of affinor metric structure when the manifold has an arbitrary dimension and the rank of the outer 2-form $d\alpha$ has any admissible value. An important special case of Legendrian submanifolds is the so called sub-Lagrangian submanifolds. Sub-Lagrangian submanifolds were introduced and described in [4]. This work extends the class of submanifolds studied in [4] to Legendrian submanifolds. Some results for the general case of affinor metric structure coincide with the well-known results for contact metric structures, and some results are specific for the case when the rank of the radical of the 2-form $d\alpha$ is higher than unity.

Legendrian submanifolds have a broad practical meaning; they are used in mechanics, fluid dynamics, physics of magnetic fields, and many other areas, where metric structures arise whose 2-form is the outer differential of the 1-form generated by a vector field. Let us consider them in more details. Let V be a vector field on a manifold M, $(.,.)$ be the scalar product of vector fields on the manifold M, and dr be a vector field of infinitesimal displacements on M. Then on M the 1-form is defined:

$$\alpha = (V, dr),$$

which generates the outer 2-form $d\alpha$ by associating this 2-form with the given scalar product by means of the affinor Φ; we obtain an affinor metric structure on the manifold M. Now, to apply different formulas expressing the values of physical characteristics of the vector field V in a bounded domain through the value at the boundary of this domain, in particular, the Stokes formula, the formula of the work of a force along a curve, we need to consider submanifolds tangent at all their points of the working distribution for the obtained affinor metric structure, because on such submanifolds the 2-form $d\alpha$ can be different from the identically zero form. This leads to the study of Legendrian submanifolds.

Section 1 provides the main definitions and information for affinor metric structures. In Section 2 we separately consider the main important special case of affinor metric structures—strict affinor metric structures. In Section 3 we provide a definition of Legendrian submanifold and also present important properties and examples of Legendrian submanifolds. In Section 4 we consider an important special case of Legendrian submanifolds—Legendrian curves. And, finally, in Section 5 we consider homogeneous Legendrian submanifolds. Many definitions and results in this work use materials from works [2–4].

1. AFFINOR METRIC STRUCTURES

In this section we provide the necessary notions and information about affinor metric structures relying upon paper [3].

Let M be a manifold of the class C^∞ and let α be a nonclosed 1-form of the class C^∞ on M.

Definition 1. The radical of a 1-form α at a point $x \in M$ is the tangent subspace

$$\operatorname{rad}\alpha_x = \{v \in T_xM : \mathrm{I}_v d\alpha_x = 0\},$$

where $d\alpha$ is the outer differential of the 1-form α and $\mathrm{I}_v d\alpha$ is the inner product of the vector v and the 2-form $d\alpha$. The radical of a 1-form α on a manifold M is the distribution of tangent subspaces

$$\operatorname{rad}\alpha = \bigcup_{x\in M} \operatorname{rad}\alpha_x;$$

a 1-form α is called regular if the distribution $\operatorname{rad}\alpha$ is regular, that is, has the same rank at all points from M.

It follows from this definition that when a 1-form α is closed, $\operatorname{rad}\alpha = TM$, and when $\operatorname{rad}\alpha = \{0\}$, $d\alpha$ is the exact symplectic form on M. In [2], we obtained the following key result for the rank of radical of a regular 1-form.

Theorem 1. *Let α be a nonclosed regular 1-form with the radical of rank r on a manifold M of dimension $n \geqslant 3$. Then*

- *If n is even, then r is also even and*

$$0 \leqslant r \leqslant n - 2.$$

- *If n is odd, then r is also odd and*

$$1 \leqslant r \leqslant n - 2.$$

The working distribution for a regular 1-form α on a manifold M is a distribution of tangent subspaces D complement to $\operatorname{rad}\alpha$ such that the restriction of the 2-form $d\alpha$ to the sections of the distribution D is a nondegenerate 2-form. Because the difference of any numbers of the same evenness always is an even number, from Theorem 2.2 we obtain the fact that the rank of the working distribution on a manifold of any dimension is always even. This allows defining the notion of affinor, which generalizes the notion of almost complex structure on a manifold.

Definition 2. Let α be a regular nonclosed 1-form on a manifold M, D be the working distribution for α, and g be the Riemannian metric on M. The affinor associated with the 1-form α on the manifold M is a continuous field of endomorphisms Φ of tangent spaces on M such that

$$d\alpha(X,Y) = g(\Phi X, Y), \quad X, Y \in C^\infty(TM),$$
$$g(\Phi X, \Phi Y) = g(X,Y), \quad X, Y \in C^\infty(D).$$

From this definition we can obtain the following properties of the affinor [3].

Proposition 1. *Let* α *be a regular nonclosed* 1*-form on a manifold* M *with the working distribution* D, Φ *be the affinor associated with a* 1*-form* α, *and id be the field of identical operators on* M. *Then*

- $\ker\Phi = \operatorname{rad}\alpha$.
- $\Phi^2\big|_D = -\operatorname{id}$.
- $\Phi^*(d\alpha) = d\alpha\circ\Phi = d\alpha$.
- $d\alpha(X,\Phi X) \geqslant 0$, $X \in C^\infty(TM)$.

These properties are sometimes taken as a definition of the affinor without taking into account the Riemannian metric. As seen in Proposition 1, the affinor is a complex structure in fibers of the working distribution D, and at $\operatorname{rad}\alpha = \{0\}$ Φ is an almost complex structure on a manifold. Now, we can define the notion of affinor metric structure.

Definition 3. The affinor metric structure on a manifold M of dimension $\geqslant 3$ is the quadruple (α, D, Φ, g), where α is a nonclosed regular 1-form on the manifold M, D is the working distribution for the 1-form α, Φ is the affinor associated with the 1-form α, and g is the Riemannian metric on M.

Remark 1. It follows from Definition 2 and Proposition 1 that for the affinor metric structure (α, D, Φ, g) the distributions D and $\operatorname{rad}\alpha$ are orthogonal with respect to the Riemannian metric g. And the affinor Φ is an orthogonal complex structure in fibers of the working distribution D.

The contact metric structure on an odd-dimensional manifold is an example of the affinor metric structure with the radical of rank 1, and Kähler structure with the exact fundamental 2-form on an even-dimensional manifold is an example of the affinor metric structure with the radical of rank 0. An example of the affinor metric structure with the radical of rank maximally possible by Theorem 1 can be found in [3]. In [2], we proved that the radical of the affinor metric structure always is the involutive distribution. Hence, by Frobenius' theorem, the radical of an affinor metric structure always is a completely holonomic distribution of tangent subspaces. The working distribution of an affinor metric structure with a nontrivial radical can be either a holonomic or nonholonomic distribution. In [4], we provided the conditions under which the working distribution is involutive and, therefore, is completely holonomic. Now, we obtain the condition under which the working distribution cannot be involutive.

Proposition 2. *Let* (α, D, Φ, g) *be an affinor metric structure on a manifold* M *of dimension* $\geqslant 3$ *and* $D \subseteq \ker\alpha$. *Then the distribution* D *is noninvolutive.*

Proof. Let $[X,Y]$ be the Lie bracket of vector fields X and Y on the manifold M. Assume that for any $X, Y \in C^\infty(D)$ we have $[X,Y] \in C^\infty(D)$. Using the invariant definition of the outer differential [5] and the condition $D \subseteq \ker\alpha$, we get

$$2d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) = 0.$$

However, it follows from Definition 3 that the 1-form α is not closed. Hence, the distribution cannot be involutive and, therefore, cannot be holonomic.

□

In what follows, we describe the class of affinor metric structures having the property given in Proposition 2.

2. CHARACTERISTIC VECTOR FIELD

Here, we describe a characteristic vector field generalizing the Reeb field [1] and class of the so called strict affinor metric structures associated with this field.

Let (α, D, Φ, g) be an affinor metric structure on a manifold M of class C^∞. By the Riesz theorem about linear functional [6] on M, there exists a unique vector field $\xi : \alpha = \mathrm{I}_\xi g$, where $\mathrm{I}_\xi g$ is the inner product, as in Definition 1. This vector field is called the characteristic vector field of the affinor metric structure. The definition immediately leads to the following properties of the characteristic vector field.

Proposition 3. *Let* ξ *be a characteristic vector field of an affinor metric structure* (α, D, Φ, g) *on a manifold* M. *Then*

- $\alpha(\xi) = g(\xi,\xi) = |\xi|^2$.
- *The distribution* $\ker\alpha$ *has a constant rank at all points from* M *iff the vector field* $\xi \neq 0$ *everywhere on* M.

- $\alpha(\Phi X) = -d\alpha(\xi, X)$, $X \in C^{\infty}(TM)$.
- *The vector field* ξ *is orthogonal to the distribution* D *with respect to the Riemannian metric* g.

Definition 4. The strict affinor metric structure is an affinor metric structure (α, D, Φ, g) with a characteristic vector field ξ such that $\xi \in C^{\infty}(\operatorname{rad}\alpha)$.

Using this definition, Proposition 3, and the definition of the Lie derivative of the 1-form in the direction of the vector field [5], we get the following result.

Theorem 2. *Let* (α, D, Φ, g) *be an affinor metric structure with a characteristic vector field* ξ. *Then the following statements are equivalent*:

- (α, D, Φ, g) *is a strict affinor metric structure*.
- $\Phi\xi = 0$.
- $\mathrm{L}_{\xi}\alpha = d|\xi|^2$, *where* $\mathrm{L}_{\xi}\alpha$ *is the Lie derivative of the* 1*-form* α *along the vector field* ξ.

From clause (3) of this theorem we have

Corollary 1. An affinor metric structure (α, D, Φ, g) with a characteristic vector field ξ of constant length is strict iff $\mathrm{L}_{\xi}\alpha = 0$.

The simplest example of a strict affinor metric structure is the contact metric structure on a odd-dimensional manifold. In this case the characteristic vector field generates the radical of rank 1, and the working distribution coincides with the contact distribution and, therefore, is not holonomic [1]. Now, we are going to generalize this fact to arbitrary strict affinor metric structures.

Proposition 4. *The working distribution of a strict affinor metric structure with the radical of rank* $\geqslant 1$ *on a manifold* M *of dimension* $\geqslant 3$ *is noninvolutive and*, *therefore*, *nonholonomic*.

Proof. Let (α, D, Φ, g) be a strict affinor metric structure with a nontrivial radical and a characteristic vector field ξ on a manifold M. It follows from Remark 1 that $\operatorname{rad}\alpha$ is orthogonal to D with respect to the Riemannian metric g, and clause (4) of Proposition 3 implies that ξ is orthogonal to $\ker\alpha$. Because $\xi \in C^{\infty}(\operatorname{rad}\alpha)$, we get $D \subseteq \ker\alpha$. Now, the statement follows from Proposition 2.

□

Proposition 5. *An affinor metric structure* (α, D, Φ, g) *with the radical of rank* $\geqslant 1$ *is strict iff* $D \subseteq \ker\alpha$.

Proof. Let (α, D, Φ, g) be a strict affinor metric structure and ξ be its characteristic vector field. Similarly to the proof of Proposition 4, we get $D \subseteq \ker\alpha$.

Conversely, if $D \subseteq \ker\alpha$, then clause (4) of Proposition 3 implies that ξ is orthogonal to D with respect to the Riemannian metric g. Now, Remark 1 leads to $\xi \in C^{\infty}(\operatorname{rad}\alpha)$.

□

Remark 2. A simple class of examples of manifolds with a nonstrict affinor metric structure (α, D, Φ, g) with a characteristic vector field $\xi : \xi \in C^{\infty}(D)$ is given by the direct product of manifolds $P \times Q$, where $\alpha|_P \neq 0$ and $TQ \subset \ker\alpha$.

Let $e(M)$ be the Euler class of a manifold M. If on M there exists a vector field different from zero at each point from M, then $e(M) = 0$ [7]. Hence, we obtain the necessary condition of existence of the affinorm metric structure.

Proposition 6. *If a real manifold* M *of dimension* $\geqslant 3$ *admits an affinor metric structure with a characteristic vector field different from zero everywhere on* M, *then the Euler class of the manifold* M *is zero*.

If M is a compact manifold without edge and $\chi(M)$ is its Euler characteristic, then $\chi(M) = 0$ [7]. This leads to

Corollary 2. If a real compact manifold without edge of dimension $\geqslant 3$ admits an affinor metric structure with a characteristic vector field different from zero everywhere on M, then its Euler characteristic is zero.

See more details about the necessary topological conditions of existence of the affinor metric structure in [3].

3. LEGENDRIAN SUBMANIFOLDS

Here, we provide the main properties and examples of Legendrian submanifolds generalizing this notion for an arbitrary affinor metric structure. Because for an affinor metric structure on a manifold with involutive working distribution, an integral submanifold passes through each point by Frobenius' theorem, a nontrivial case for studying submanifolds tangent to the working distribution is the case when the working distribution is noninvolutive.

Let (α, D, Φ, g) be an affinor metric structure with a nontrivial radical and a noninvolutive working distribution D on a manifold M of class C^∞. The Legendrian submanifold for this affinor metric structure is a submanifold $Q : TQ \subset D|_Q$. Note that for subtwistor structures with exact fundamental 2-form, the sub-Lagrangin submanifolds introduced in [4] always are Legendrian submanifolds. In addition to that, any curve $\gamma(t) : \dot\gamma(t) \in C^\infty(D)$, where $\dot\gamma(t)$ is a tangent vector to the curve $\gamma(t)$ is a Legendrian submanifold of dimension 1. Proposition 4 implies that the working distribution of a strict affinor metric structure always is noninvolutive. Now, we are going to obtain the upper bound on the dimension of Legendrian submanifolds for strict affinor metric structures.

Proposition 7. *Let (α, D, Φ, g) be a strict affinor metric structure with a radical of rank $r \geqslant 1$ on a manifold M of dimension $n \geqslant 3$. Then the dimension of any Legendrian submanifold in M is $\leqslant \frac{n-r}{2}$.*

Proof. Let Q be a Legendrian submanifold in M. Proposition 5 leads to $TQ \subset D \subseteq \ker\alpha$. Because the tangent distribution TQ is an involutive distribution on Q, from the definition of the outer differential of a 1-form, for any $X, Y \in C^\infty(TQ)$ we obtain

$$2d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) = 0,$$

and from Definition 2 we have

$$g(\Phi X, Y) = d\alpha(X,Y) = 0,$$

that is, the distributions TQ and $\Phi(TQ)$ are orthogonal. Because the affinor Φ is an automorphism of fibers of the working distribution and $\operatorname{rank}(\Phi(TQ)) = \operatorname{rank}(TQ)$, we get

$$2\operatorname{rank}(TQ) = \operatorname{rank}(TQ \oplus \Phi(TQ)) \leqslant \operatorname{rank}(D) = n - r.$$

□

The proof of Proposition 7 results in

Corollary 3. Let (α, D, Φ, g) be a strict affinor metric structure on a manifold M of dimension $\geqslant 3$. Then $Q\ d\alpha|_Q = 0$ for any Legendrian submanifold.

This corollary shows that a Legendrian submanifold of maximally possible dimension for a strict metric structure is a sub-Lagrangian submanifold for the corresponding subtwistor structure [4]. In addition, it shows that a restriction of a 2-form $d\alpha$ to a Legendrian submanifold of even dimension cannot be a symplectic structure. Now, we are going to demonstrate that a restriction of an affinor Φ to a Legendrian submanifold of even dimension also cannot be an almost complex structure.

Proposition 8. *Let (α, D, Φ, g) be a strict affinor metric structure on a manifold M. Then for any even-dimensional Legendrian submanifold Q a restriction of the affinor Φ on Q cannot be an almost complex structure on Q.*

Proof. Let Q be an even-dimensional Legenrian submanifold in M. Assume that a restriction of the affinor Φ on Q is an almost complex structure on the submanifold Q. The definition of the almost complex structure on a manifold ([5], Chapter 9) implies that the tangent distribution TQ is a distribution on Q invariant under the action of the field of endomorphisms Φ. However, the proof of Proposition 7 implies that the distribution $\Phi(TQ)$ is orthogonal to the distribution TQ with respect to the Riemannian metric g. Hence, a restriction of the affinor Φ on Q cannot be an almost complex structure.

□

Remark 3. All results obtained above for strict affinor metric structures remain valid in the case when (α, D, Φ, g) is a nonstrict affinor metric structure with a nontrivial radical and Q is a Legendrian submanifold such that $TQ \subset (D \cap \ker\alpha)|_Q$.

Now, we provide some examples of Legendrian submanifolds.

Example 1. Consider a direct product $M = S^{2n+1} \times T^{r-1}$, where S^{2n+1} is a sphere of dimension $2n+1$ and T^{r-1} is a torus of dimension $r-1$. The group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ acts on S^{2n+1} and generates a vector field ξ on S^{2n+1} tangent to the orbit of this action. Let g_0 be a Riemannian metric on the sphere S^{2n+1}. The vector field ξ generates a regular 1-form $\alpha = \mathrm{I}_\xi g_0$ on S^{2n+1}. In [8], Boothby and Wang showed that $\operatorname{rad} \alpha = \mathbb{R}\xi$ and the distribution $D = \ker \alpha$ is noninvolutive. Because the sphere S^{2n+1} is the principal distribution over the complex projective space $\mathbb{C}P^n$ with the fiber S^1, the manifold M is the principal distribution over $\mathbb{C}P^n$ with the fiber $T^r = S^1 \times T^{r-1}$, and the distribution D is the connectivity on this distribution. Let us continue the 1-form α on M assuming $\alpha|_{T^r} = 0$. Let $g = g_0 + g_1$ be a Riemannian metric on M, where g_1 is a Riemannian metric on the torus T^{r-1}. In [4], we demonstrated that a complex structure in the space $\mathbb{C}P^n$ generates an affinor Φ on the principal distribution M associated with the 1-form α. We have obtained a strict affinor metric structure (α, D, Φ, g) on M with a radical $T(T^r)$ of rank r. The space $\mathbb{C}P^n$ contains a submanifold S of real dimension n:

$$S = \left\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \backslash \{0\} : \operatorname{im}(z_k) = \operatorname{re}(z_k), \quad k = 0, 1, \ldots, n\right\}.$$

Let π be the projection $S^{2n+1} \to \mathbb{C}P^n$. In [4], we showed that $\pi^{-1}(S)$ is a sub-Lagrangian submanifold of dimension n in M. Now, we obtain the fact that for any submanifold $Q \subseteq \pi^{-1}(S)$ there exists a Legendrian submanifold in M.

Example 2. Let P be the principal distribution over an Hermitian manifold M with a structural Abelian group G of dimension r, for example, G is a torus of dimension r. Let D be a connectivity on the principal distribution P, ω be a form of connectivity, and Ω be a form of curvature of connectivity D such that $\Omega \circ d\pi = \Omega_0$, where Ω_0 is the fundamental 2-form of the Hermitian structure on M and π is the projection $P \to M$. This implies that the 2-form Ω is not degenerate on sections of the distribution D. The latter condition means that at least one of the coordinate 1-forms ω_k, $1 \leqslant k \leqslant r$, is not closed and is not degenerate on the sections of the distribution D. We put $\alpha = \omega_k$. Because the Lie algebra of the group G is commutative, the structural equation for the form of curvature of connectivity is given by $\Omega = d\omega$. Because $\ker(d\pi) = TG$ and the 2-form $\Omega = d\omega$ is not degenerate on sections of the distribution D, it is true that $\operatorname{rad} \omega = TG$, and, therefore, $\operatorname{rad} \alpha = TG$ also holds. The Hermitian metric on the base M and the Riemannian metric on fibers of the principal distribution generate a Riemannian metric g on the manifold P. As in Example 1, the complex structure on the base M generates an affinor Φ on P. We have obtained an affinor metric structure (α, D, Φ, g) with a radical of rank r on P. The definition of the form of connectivity implies that $D \subset \ker \alpha$. From Proposition 5 we immediately obtain the fact that this affinor metric structure is strict.

For each point $x \in P$, by Γ_x we denote the set of all curves arising from the point x such that for any curve $\gamma(t) \in \Gamma_x$ the tangent vector $\dot{\gamma}(t) \in D|_{\gamma(t)}$ for any value of the parameter t. Then any submanifold $Q_x \subset \Gamma_x$ is a Legendrian submanifold in P. In addition, Proposition 7 implies that Q_x cannot coincide with Γ_x.

Example 3. Let (α, D, Φ, g) be a nonstrict affinor metric structure on a manifold M of dimension $\geqslant 3$ with a characteristic vector field ξ. Then the projection Z of the vector field ξ on the working distribution D is an everywhere nonzero vector field on M such that $Z \in C^\infty(D)$. Because the affinor Φ is an orthogonal complex structure in fibers of the working distribution D, the vector fields Z and ΦZ are orthogonal with respect to the metric g at each point from M. If the Lie bracket $[Z, \Phi Z] = \lambda Z$, where $\lambda \in C^\infty(M)$, these vector fields generate an involutive distribution K of rank two. By Frobenius' theorem, an integral submanifold $Q_x : TQ_x = K|_{Q_x}$ passes through each point $x \in M$. Because any almost complex structure is integrable on a two-dimensional manifold ([5], Chapter 9), the restriction of the affinor Φ on the submanifold Q_x is a complex structure on Q_x. Now, we obtain the fact that a Legendrian complex curve Q_x passes through each point $x \in M$, and the restriction of the affinor metric structure on Q_x induces a Kählerian structure on Q_x.

Remark 4. We can generalize Example 3 if we consider an arbitrary affinor metric structure with a nontrivial radical. On the manifold M, instead of the characteristic vector field, we consider an arbitrary vector field whose projection onto the working distribution is different from zero everywhere on M. However, if the characteristic vector field is different from zero at each point from M, then other vector fields different from zero not always can exist on M.

4. LEGENDRIAN CURVES

In this section, we consider one-dimensional Legendrian submanifolds, that is, Legendrian curves.

Let (α, D, Φ, g) be an affinor metric structure with a nontrivial radical on a manifold M of class C^∞. It is clear that any vector field $X \in C^\infty(D)$ in the neighborhood of each point $x \in M$ generates a Legendrian curve in M. In addition to that, a Legendrian curve in the neighborhood of each point is generated by any vector field on M whose projection onto the working distribution is nonzero. In addition, any curve lying in a Legendrian submanifold is a Legendrian curve. In particular, the boundary of a Legendrian two-dimensional submanifold with an edge is a Legendrian curve. From Example 3 and Remark 4 we get

Corollary 4. Let (α, D, Φ, g) be a nonstrict affinor metric structure with a nontrivial radical on a manifold M and ξ be the projection of its characteristic vector field on the working distribution D. Then

- The vector fields ξ and $\Phi\xi$ generate Legendrian curves arising from the point x orthogonal with respect to the Riemannian metric g in the neighborhood of each point $x \in M$.
- If the vector fields ξ and $\Phi\xi$ generate an involutive distribution on M, then a Legendrian complex curve Q_x passes through each point $x \in M$.

Remark 5. Proposition 8 and the properties of an affinor in Proposition 1 imply that for a strict affinor metric structure (α, D, Φ, g) on a manifold M, for any vector field $X \in C^\infty(D)$ different from zero at each point from M, the vector fields X and ΦX generate a pair of orthogonal Legendrian curves arising from the point x in the neighborhood of each point $x \in M$, but do not generate a complex Legendrian curve arising from the point x.

Let (α, D, Φ, g) be an affinor metric structure on a manifold M of dimension $n \geqslant 3$ with a noninvolutive working distribution and a radical of rank $r \geqslant 1$. Then, in the neighborhood of each point $x \in M$, there exists a local basis $e_1, \ldots, e_r$ of the distribution $\operatorname{rad}\alpha$ orthonormal with respect to the metric g. We obtain from Remark 1 the fact that in some neighborhood of the point x, any smooth Legendrian curve $\gamma(t)$ with the origin at the point x is given by a system of ordinary differential equations

$$g(\dot\gamma(t), e_k) = 0, \quad k = 1, 2, \ldots, r.$$

If we supplement this basis to the orthonormal local basis of the tangent distribution TM assuming that the latter r basis vector fields are $e_1, \ldots, e_r$, then we get

$$x_{n-r+1}(t) = x_{n-r+2}(t) = \ldots = x_n(t) = 0,$$

where $x_k(t)$ is the kth coordinate of the curve $\gamma(t)$. This leads to

Proposition 9. *Let* (α, D, Φ, g) *be an affinor metric structure with a noninvolutive working distribution and a radical of rank* $r \geqslant 1$ *on a manifold* M *of dimension* $n \geqslant 3$. *Then for any point* $x \in M$ *and for any smooth Legendrian curve* $\gamma(t) : \gamma(0) = x$, *there exists an open neighborhood* U *and local coordinates* $(x_1, \ldots, x_n)$ *in the neighborhood* U *in which the curve* $\gamma(t)$ *has the form* $(x_1(t), \ldots, x_{n-r}(t), 0, \ldots, 0)$.

Proposition 9 implies that the union of all Legendrian curves for an affinor metric structure with a noninvolutive working distribution arising from a single point forms a space of dimension $n - r$, and Proposition 7 implies that this space cannot be a manifold. Thus, we get

Corollary 5. The union of all Legendrian curves arising from a common point for any affinor metric structure with a noninvolutive working distribution D does not admit the structure of a manifold. However, it always contains a Legendrian submanifold of dimension k, $1 \leqslant k \leqslant \dfrac{\operatorname{rank}(D)}{2}$.

Remark 6. Because the working distribution of an affinor metric structure with zero radical on a manifold M coincides with the tangent distribution TM, Corollary 5 implies that for any affinor metric structure with a noninvolutive working distribution, for instance, for a strict affinor metric structure, the union of all Legendrian curves arising from a common point forms a proper subset of the manifold M that is not a submanifold.

Let $\gamma_1(t)$, $t \in [0,1]$, $x = \gamma_1(0) = \gamma_1(1)$, be a Legendrian loop on a manifold M and suppose that $\gamma_2(t)$, $x = \gamma_2(0) = \gamma_2(1)$, is a loop homotopic to it on M. Let us study the question when the loop $\gamma_2(t)$ also is a Legendrian curve.

Proposition 10. *Let* (α, D, Φ, g) *be an affinor metric structure on a manifold* M *of dimension* $\geqslant 3$, $\gamma(t)$ *be a nontrivial Legendrian smooth oriented loop on* M *with an origin at a point* x, *and* $\dot{\gamma}(t)$ *be a velocity vector field of the curve* $\gamma(t)$. *Then the vector field* $\Phi\dot{\gamma}(t)$ *generates a Legendrian loop with the origin at the point* x *that is homotopic to the loop* $\gamma(t)$.

Proof. By *id* we denote the field of identical linear operators on M. Consider a one-parametric family of fields of endomorphisms of tangent spaces

$$\Phi_s = s\Phi + (1-s)\,\mathrm{id},\ s \in [0,1].$$

The Riemannian metric g from the affinor metric structure allows defining the geodesics and the exponential mapping $\exp : T_xM \to M$ on M. Then the mapping $F(s,t) = \exp(\Phi_s\dot{\gamma}(t))$ is a continuous mapping $[0,1]\times[0,1] \to M$. Here, for any s

$$F(s,0) = \exp(\Phi_s\dot{\gamma}(0)) = \exp(\Phi_s\dot{\gamma}(1)) = F(s,1)$$

and $\frac{d}{dt}F(1,t) = \Phi\dot{\gamma}(t) \in C^\infty(D)$. We obtain the fact that the curve $\gamma_1(t) = F(1,t)$ is a Legendrian loop with the origin at the point $x = \gamma(0)$ homotopic to the Legendrian loop $\gamma(t)$.

□

Let (α, D, Φ, g) be an affinor metric structure on a manifold M with a noninvolutive working distribution D. Note that the trivial loop $\gamma(t) = x \in M$ for any $t \in [0,1]$ has a zero velocity vector field belonging to $C^\infty(D)$, and, consequently, is a Legendrian curve. Let $\Pi_1(M)$ be the first fundamental group of the manifold M. We can easily check that the Legendrian loops from $\Pi_1(M)$ form a subgroup. We denote this subgroup by $\Pi L_1(M)$. The subgroup consists of classes of homotopic Legendrian loops on M, that is, an element of this subgroup is the class of all Legendrian loops homotopic to some Legendrian loop. Now, we arrive at

Proposition 11. *If a manifold* M *admits an affinorm metric structure with a noninvolutive working distribution, in particular, a strict affinor metric structure, then any loop on* M *is homotopic to a Legendrian loop iff* $\Pi_1(M) = \Pi L_1(M)$.

Corollary 6. If a manifold M admits an affinor metric structure with a noninvolutive working distribution for which the subgroup $\Pi L_1(M)$ consists only of the trivial loop, then any Legendrian loop on M is contractable, that is, is homotopic to the point.

The index of a normal subgroup H in a group $\Pi_1(M)$ is the number of generatrices in $\Pi_1(M)$ generating different cosets by the subgroup H. The group $\Pi_1(M)$ freely acts on the universal covering $\tilde{M}$ for the manifold M, and the factor $\tilde{M}$ by the action of the subgroup H is a kth-sheet covering of the manifold M, where k is the index of the subgroup H (see more details in [9]). Hence, we obtain

Proposition 12. *If a manifold* M *admits an affinor metric structure with a noninvolutive working distribution, in particular, a strict affinor metric structure, and* $\Pi L_1(M)$ *is a normal subgroup of index* k *in* $\Pi_1(M)$, *then for the manifold* M *there exists a* k*-sheet covering. At* $k = 1$ M *can be not simply connected, but any loop on* M *is homotopic to a Legendrian loop.*

Remark 7. Proposition 10 allows introducing the equivalence relation for tangent vectors to Legendrian loops for an affinor metric structure (α, D, Φ, g) on a manifold M, assuming that the tangent vectors v and w are equivalent if $w = \Phi v$. After factorization by this equivalence relation, the dimension of the tangent space for the subgroup $\Pi L_1(M)$ at the origin is no higher than $\frac{\mathrm{rank}(D)}{2}$.

5. HOMOGENEOUS LEGENDRIAN SUBMANIFOLDS

Homogeneous spaces and Lie groups are important classes of manifolds; therefore, we consider them separately in this section. On such manifolds we can define a special class of affinor metric structures for which all properties are independent of the point of manifold and must be investigated only at one of its points.

Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, and A be a linear automorphism of the Lie group $\mathfrak{g}$. An affinor metric structure (α, D, Φ, B) on the Lie group G is called A-invariant if $\alpha \circ A = \alpha$, $B \circ A = B$, $\Phi \circ A = A \circ \Phi$ and the subspace D is invariant with respect to the action of the automorphism A. A structure is called left-invariant if A is the differential of the left shift mapping $L_g : L_g(x) = gx$, $g \in G$, right-invariant if A is the differential of the right shift mapping $R_g : R_g(x) = xg$, $g \in G$, and bi-invariant if A is the differential of the inner automorphism of the group G $A_g : A_g(x) = gxg^{-1}$, $g \in G$. For a left- or right-invariant affinor metric structure, its value at any point $g \in G$ coincides with its value at the unity of the group e; therefore, it suffices to consider the affinor metric structure $(\alpha_e, D_e, \Phi_e, B_e)$ on the Lie algebra $\mathfrak{g}$. In what follows, we use the notation $Ad_g = dA_g = dL_g \circ dr_{g^{-1}}$.

Let $M = G/H$ be a homogeneous space, where G is a connected Lie group acting on a manifold M effectively and transitively, H is a connected subgroup of isotropy of the origin O, and $\mathfrak{h}$ is the Lie algebra of the subgroup of isotropy H. A left-invariant affinor metric structure (α, D, Φ, B) on the Lie group G is called isotropically degenerate if $\mathfrak{h} \subseteq \operatorname{rad}\alpha$. An affinor metric structure on a homogeneous manifold $M = G/H$ is called G-invariant if it is A-invariant and A is the differential of the mapping $g : M \to M$, $g \in G$. In [10], we showed that the G-invariant affinor metric structure on a homogeneous space $M = G/H$ raises to the G-left-invariant H-right-invariant isotropically degenerate affinor metric structure on the Lie group G and, conversely, the G-left-invariant H-right-invariant isotropically degenerate affinor metric structure on the Lie group G induces a G-invariant affinor metric structure on the homogeneous space $M = G/H$. Here, the Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$, $\mathfrak{h}$ is the Lie algebra of the subgroup of isotropy H, and $\mathfrak{p}$ is the Ad_H-invariant subspace orthogonal to $\mathfrak{h}$ with respect to the left-invariant metric B. In addition, the definition of the G-invariant affinor metric structure on a homogeneous space $M = G/H$ implies that the Lie group G is the group of isometries of the Riemannian metric B and H is a compact subgroup isomorphic to the subgroup of orthogonal linear operators in the vector space D_O (see more details in [5]). Because the G-invariant 1-form on a homogeneous space $M = G/H$ is different from zero at all points, the characteristic vector field of G-invariant affinor metric structure is everywhere different from zero on the homogeneous space M. Because any homogeneous space is a manifold without edge, Proposition 6 and Corollary 2 lead to

Corollary 7. Let $M = G/H$ be a homogeneous space of dimension $\geqslant 3$. If one of the following conditions is fulfilled:

- the manifold M has a nonzero Euler class;
- M is a compact manifold with a positive Euler characteristic,

then there exists no G-invariant affinor metric structures on M.

An n-dimensional sphere S^n is a compact homogeneous space. In [10], we proved that for $n \geqslant 3$ there exists no invariant affinor metric structures on S^n. Also, in [10], we proved that a four-dimensional homogeneous space $M = G/H$ admits an invariant affinor metric structure only at $H = \{e\}$, that is, only in the case of the Lie group.

Let ξ be the characteristic vector field of a G-invariant affinor metric structure (α, D, Φ, B) on a homogeneous space $M = G/H$. Because G is a group of isometries of a Riemannian metric B, the one-parametric subgroup generated by the vector field ξ also is a group of isometries of this metric, that is, ξ is a Killing vector field. An affinor metric structure with a Killing characteristic vector field is called the K-affinor metric structure. As we see, any invariant affinor metric structure is K-affinor. Joining the results from Section 3 and the results proved in [3] for K-affinor metric structures, we get the following statement.

Theorem 3. *Let* (α, D, Φ, B) *be a* G*-invariant affinor metric structure with a characteristic vector field* ξ *on a homogeneous space* $M = G/H$, ∇ *be a covariant derivative for the Levi-Civita connectivity of the Riemannian metric* B, *and* $\mathrm{L}_\xi\alpha$ *be the Lie derivative of the* 1*-form* α *along the vector field* ξ. *Then the following conditions are equivalent*:

- $\xi \in \operatorname{rad}\alpha$.
- $D \subseteq \ker\alpha$.

- $\nabla_\xi \xi = 0$.
- $\Phi = \nabla \xi$.
- $\mathrm{L}_\xi \alpha = 0$.

Proposition 13. *Let $M = G/H$ be a homogeneous space with the origin O and let (α, D, Φ, B) be a G-left-invariant H-right-invariant isotropically degenerate affinor metric structure on the Lie group G. Then M admits homogeneous Legendrian submanifolds of dimension $k \geqslant 1$ with the origin O iff the working distribution D contains a Lie subalgebra of dimension k.*

Proof. Let D_O be a fiber of the working distribution for a G-invariant affinor metric structure at the origin O obtained from the affinor metric structure (α, D, Φ, B) and e is the unity of the group G. If Π is the projection $G \to M$, then $d\Pi(D_e) = D_O$. This specifies a correspondence between the G-invariant working distribution on M and the left-invariant working distribution on G. Any homogeneous submanifold $N \subset M$ with the origin O has the form $N = K/H$, where K is a connected proper subgroup in G. Hence, $K = QH$, where Q is a subgroup in G such that its Lie algebra $\mathfrak{q} \subset \mathfrak{p}$, where $\mathfrak{p}$ is an Ad_H-invariant subspace from the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$. Because D_e is an Ad_H-invariant subspace in $\mathfrak{p}$ [10], we obtain the fact that a homogeneous submanifold N is Legendrian iff $\mathfrak{q}$ is a subalgebra in D_e. In addition to that, the construction of the subgroup Q leads to $\dim(N) = \dim(\mathfrak{q})$.

□

Proposition 7 and Theorem 3 lead to

Corollary 8. Let (α, D, Φ, B) be a G-invariant affinor metric structure with a radical of rank $r \geqslant 1$ on a homogeneous space $M = G/H$ of dimension $n \geqslant 5$ with the origin O. If one of the conditions of Theorem 3 is fulfilled, then the dimension of any homogeneous Legendrian submanifold with the origin O is no higher than $\frac{n-r}{2}$.

Example 4. Let $M = G/H$ be a homogeneous space of dimension $\geqslant 5$ and (α, D, Φ, B) be a G-left-invariant H-right-invariant affinor metric structure on the Lie group G. Explicit examples of such structures can be found in [2, 10]. Each left-invariant vector field $X \in C^\infty(D)$ generates a one-parametric subgroup $G_X(t) \subset G$. Let $\mathfrak{q}$ be a subalgebra of dimension k in D. In $\mathfrak{q}$ we choose a basis $e_1, \ldots, e_k$ that generates transversal one-parametric subgroups $G_1(t_1), \ldots, G_k(t_k)$ in G. Then $Q = \prod_{j=1}^{k} G_j(t_j)$ is a k-dimensional subgroup in G with the Lie algebra $\mathfrak{q} \subset D$. By Proposition 13 we obtain the fact that $N = Q/H$ is a Legendrian homogeneous submanifold of dimension k with the origin O in M.

Remark 8. Theorem 1 implies that for any affinor metric structure with a nontrivial noninvolutive working distribution on a manifold of dimension $\leqslant 4$, the working distribution has only rank two. Hence, Legendrian submanifolds of dimension higher than unity cannot exist on manifolds of dimension $\leqslant 4$.

Let $M = G/H$ be a symmetric space of dimension $\geqslant 5$. For a symmetric space the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ has the properties

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}, \quad [\mathfrak{p}, \mathfrak{h}] \subseteq \mathfrak{p}.$$

Hence, the subspace $\mathfrak{p}$ contains no subalgebras of dimension higher than unity. Now, from Proposition 13 we get

Corollary 9. If a symmetric space M admits an invariant affinor metric structure, then in M there exist no homogeneous Legendrian submanifolds of dimension $\geqslant 1$.

Thus, the symmetric space of any dimension is an example of homogeneous space not admitting Legendrian homogeneous submanifolds of dimension higher than unity.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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